Uniform approximation by Fourier sums on the classes of differentiable functions of higher smoothness

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Let C and L_p, $1 \le p \le \infty$, be the spaces of 2π–periodic functions with the standard norms $\|\cdot\|_C$ and $\|\cdot\|_p$. Let $r > 1$ and $\beta \in \mathbb{R}$. Further, let $W^r_{\beta,p}$ be the set of all 2π-periodic functions f, representable as convolutions of the form

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi(x-t)B_{r,\beta}(t)dt, \quad a_0 \in \mathbb{R}, \quad \|\varphi\|_p \le 1, \quad \varphi \perp 1,$$

where $B_{r,\beta}(t)$ are Weyl-Nagy kernels of the form

$$B_{r,\beta}(t) = \sum_{k=1}^{\infty} k^{-r}\cos\left(kt - \frac{\beta\pi}{2}\right).$$

If $r \in \mathbb{N}$ and $\beta = r$, then the classes $W^r_{\beta,p}$ coincide with the well-known classes W^r_p, which consist of 2π-periodic functions with absolutely continuous derivatives up to $(r-1)$-th order inclusive and such that $\|f^{(r)}\|_p \le 1$.

We investigate the quantity $\mathcal{E}_n(W^r_{\beta,p})_C = \sup\limits_{f\in W^r_{\beta,p}} \|f - S_{n-1}(f)\|_C$, where $S_{n-1}(f)$ is the partial Fourier sum of order $n-1$ of f.

Theorem 1. *Let $1 \le p \le \infty$, $n \in \mathbb{N}$ and $\beta \in \mathbb{R}$. Then for $r \ge n+1$ the following estimate is true*

$$\mathcal{E}_n(W^r_{\beta,p})_C = \frac{1}{n^r}\left(\frac{\|\cos t\|_{p'}}{\pi} + O(1)\left(1+\frac{1}{n}\right)^{-r}\right), \qquad (1)$$

where $\frac{1}{p}+\frac{1}{p'}=1$, $O(1)$ is quantity uniformly bounded in all analyzed parameters. If, in adition, $r/n \to \infty$, then formulas (1) *becomes the asymptotic equality.*

For $p = \infty$ the estimate (1) was istablished by S.B. Stechkin.