Інститут теоретичної фізики ім. М.М. Боголюбова
Національної академії наук України

Кваліфікаційна наукова
праця на правах рукопису

Журавльов Юрій Сергійович

УДК 538.9; 537.611.2

ДИСЕРТАЦІЯ

Метод ефективних форм-факторів в квантових інтегровних моделях

01.04.02 Теоретична фізика

10 Природничі науки (104 Фізика та астрономія)

Подається на здобуття наукового ступеня доктора філософії

Дисертація містить результати власних досліджень. Використання ідей, результатів і текстів інших авторів мають посилання на відповідне джерело

____

(підпис, ініціали та прізвище здобувача)

Науковий керівник — Іоргов Микола Зіновійович, доктор фіз.-мат. наук

Київ - 2024

Bogolyubov Institute for Theoretical Physics
of the National Academy of Sciences of Ukraine

Qualifying scientific work
on the rights of the manuscript

Zhuravlov Yurii Serhiiovych

UDC 538.9; 537.611.2

Dissertation

The method of effective form-factors in quantum integrable models

01.04.02 Theoretical physics

10 Natural sciences (104 Physics and astronomy)

Applying for the Doctor of Philosophy degree

The dissertation contains the results of own research. The use of ideas, results, and texts of other authors have reference on the corresponding source

____

(signature, initials and last name of the candidate)

Scientific supervisor — Iorgov Mykola Zinoviiovych,
doctor of physics and mathematics

Kyiv - 2024

АНОТАЦІЯ

Журавльов Ю.С. Метод ефективних форм-факторів в квантових інтегровних моделях. - Кваліфікаційна наукова праця на правах рукопису.

Дисертація на здобуття наукового ступеня доктора філософії за спеціальністю 01.04.02 “Теоретична фізика” (104 - Фізика та астрономія). - Інститут теоретичної фізики ім. М. М. Боголюбова Національної академії наук України, Київ, 2024.

Ключові слова: фізика конденсованого середовища, квантові інтегровні системи, спінові ланцюжки, нерівноважна фізика, транспорт в одновимірних системах, асимптотичний аналіз, метод форм-факторів.

Дисертація базується на роботах [1,2,3], де розглядалися асимптотичні властивості кореляційних функцій одновимірних квантових систем. Є кілька причин для вивчення одновимірної квантової фізики.

Перш за все, квантові одновимірні моделі завжди привертали велику увагу завдяки багатству математичних структур, які виникали при дослідженні їхніх кореляційних функцій, і завдяки можливості дослідження непертурбативних явищ. Кульмінацією цих розробок стало формулювання моделі Латтінжера, що є ефективною теорією поля при низьких температурах.

Ще одна причина є більш практичною. З розвитком експериментальних технік в екпериментах з холодними атомами було спостережено багато неочікуваних властивостей в поведінці одновимірних квантових систем. Відмітимо роботу *Kinoshita et al* (2006), в якій автори спостерігали, що для системи холодних атомів, обмежених рухатися лише в одному вимірі, початково збурена система не переходила до стану рівноваги. Така поведінка є типовою для інтегрованих систем, які мають багато інтегралів руху, що запобігають термалізації системи. Зацікавленість у нерівноважній динаміці або динаміці високо збуджених станів стимулювала багато теоретичних досліджень, що призвели до виникнення нових концепцій, таких як узагальнені ансамблі Гіббса, метод квенч-дії (quench action), узагальнена гідродинаміка та інші.

Одна з додаткових причин для дослідження одновимірних квантових систем полягає у мініатюризації електронних пристроїв. Вважається, що одного дня електричні проводи можуть стати ефективно одновимірними. Тому

варто вивчати транспорт в одновимірних системах. Одним із найбільш дивовижних явищ у цій області є квантування провідності в квантових точкових контактах. Теоретичне пояснення цього явища було зроблено за допомогою формалізму Ландауера-Бюттікера, який пов'язує матрицю проходження (transmission matrix) з провідністю. Хоча в елементарній теорії тунелювання ймовірність проходження визначається в стаціонарній постановці задачі, було також приділено багато уваги нерівноважному підходу до задачі транспорту.

Потужні аналітичні методи, що включають техніку Келдиша для функції Гріна, були розроблені різними авторами. З точки зору одновимірних інтегровних моделей, зацікавленість до подібних проблем була спричинена розглядом квантових квенчей (quantum quench). Останні виникають в задачах, в яких необхідно знайти часову еволюцію ізольованої квантової системи, яка в початковий момент часу знаходиться в високонерівноважному стані, створеному або за допомогою швидкої зміни гамільтоніана, або містить макроскопічні просторові неоднорідності.

Щодо об'єкта дослідження, увага в дисертації переважно зосереджується на кореляційних функціях. Основним підходом до обчислення кореляційних функцій в інтегровних моделях є пряме підсумовування форм-факторів у спектральному розкладі. Обчислення кореляційних функцій при скінченій температурі або, більш загально, в станах зі скінченною ентропією дуже відрізняється від вакуумного випадку через те, що форм-фактори як функції від розміру системи спадають експоненційно, а не степеневим чином. Тому були розроблені різні методи для розв'язання такого роду проблем, включаючи квантовий метод матриці переходу (transfer matrix), нелінійні диференціальні рівняння, які виводяться за допомогою задачі Рімана-Гільберта, аксіоматичне визначення теплових форм-факторів в інтегровних квантових теоріях поля, а також часткове підсумовування кількох частинково-діркових збуджень і виділення найбільш сингулярних частин форм-факторів.

Дисертаційна робота базується на трьох статтях. В [1] розглядався квантовий XY спіновий ланцюжок. Ця модель була розглянута вперше в *Lieb et al* (1961) як окремий випадок моделі XYZ спінового ланцюжка Гейзенберга ($J_x \neq J_y \neq 0, J_z = 0$). Основним посиланням для ознайомлення з цією моделлю є *Izergin et al* (2000), де автори вивели представлення кореляційної функції $G(m)$ двох спінових операторів (m — відстань між ними) у термінах різниці двох детермінантів Фредгольма. На основі цього результату нами було розроблено метод асимптотичного аналізу кореляційної функції для випадку, коли відстань m прямує до нескінченності. Метод полягає у визначені ефективних форм-факторів для одновимірних граткових ферміонів із довільними фазовими зсувами. Далі вводиться тау-функція, що є рядом по цим форм-факторам. З одного боку, було виконане точне підсумовування тау-функції в термодинамічній границі та представлено відповідь за допомогою детермінантів Фредгольма. З іншого боку, прості вирази для форм-факторів дозволили представити відповідні ряди як інтеграли від елементарних функцій. Використовуючи цей підхід, були переотримані асимптотики статичних кореляційних функцій квантового XY ланцюжка при скінченній температурі.

У [2] вивчалась модель одновимірних непроникних еніонів на гратці. Ця модель була введена *Paţu* (2015) як узагальнення моделі квантового XXO спінового ланцюжка. Автором знайдено двоточкову кореляційну функцію $G(x,t)$ у термодинамічній границі у вигляді різниці двох детермінантів Фредгольма. Для опису поведінки цієї кореляційної функції, у випадку коли час та відстань прямують до нескінченності, в дисертації був використаний підхід ефективних форм-факторів. Відстань x та час t прямували до нескінченності таким чином, щоб їх відношення залишалося постійним $x/t = \mathrm{const}$. Знайдені асимптотичні вирази відрізняються в просторово-подібній ($x/t > 1$) і часоподібній ($x/t < 1$) областях. Зокрема, у часоподібній області, окрім експоненціального спаду, спостерігався додатковий степеневий множник. Було показано, що цей результат є універсальним, оскільки він пов'язаний виключно із розривною поведінкою функції фазового зсуву. При спеціальному значенні еніонного параметра, $\kappa = 1$, була переотримана асимптотика спін-спінових динамічних кореляційних функцій у моделі квантового XXO спінового ланцюжка.

У [3] досліджувався транспорт в одновимірних системах вільних ферміонів під впливом довільних локальних потенціалів. Була розглянута система, що складається з двох частин. Умова, необхідна для початку транспорту, моделювалася початковим дуже нерівноважним розподілом, для якого, населеною була лише половина системи. На додаток до цього, локальний потенціал також раптово змінювався, коли починався рух частинок. Для такого квенч протоколу (quench protocol) було обчислено повну статистику підрахунку (Full Counting Statistics) кількості частинок $\mathcal{F}(t)$ у частині системи, що початково була порожньою. У термодинамічній границі $\mathcal{F}(t)$ була виражена через детермінант Фредгольма з ядром, яке залежить від даних розсіювання та функцій Йоста для потенціалів до квенча та після. Була досліджена асимптотика отриманого детермінанта у випадку, коли час прямує до нескінченності. Було спостережено, що якщо два або більше зв'язаних стани присутні в спектрі потенціалу після квенча, то інформація про початковий стан проявляється у вигляді стійких коливань $\mathcal{F}(t)$. Навпаки, коли зв'язаних станів немає, то асимптотична поведінка $\mathcal{F}(t)$ визначається виключно даними розсіювання потенціалу після квенча. Асимптотичний струм, порахований як перший момент $\mathcal{F}(t)$ для цього випадку, відтворює формулу Ландауера–Бюттікера.

Список публікацій:

1. O. Gamayun, N. Iorgov and Y. Zhuravlev, *Effective free-fermionic form factors and the XY spin chain*, SciPost Physics **10**(3) (2021), doi:10.21468/scipostphys.10.3.070 .

2. Y. Zhuravlev, E. Naichuk, N. Iorgov and O. Gamayun, *Large-time and long-distance asymptotics of the thermal correlators of the impenetrable anyonic lattice gas*, Physical Review B **105**(8) (2022), doi:10.1103/physrevb.105.085145 .

3. O. Gamayun, Y. Zhuravlev and N. Iorgov, *On Landauer–Büttiker formalism from a quantum quench*, Journal of Physics A: Mathematical and Theoretical **56**(20), 205203 (2023), doi:10.1088/1751-8121/accabf .

ABSTRACT

Yurii Zhuravlov The method of effective form-factors in quantum integrable models. - Manuscript.

Thesis for the Doctor of Philosophy degree in specialty 01.04.02 “Theoretical physics” (104 - Physics and astronomy). – Bogolyubov Institute for Theoretical Physics of National Academy of Sciences of Ukraine, Kyiv, 2024.

Keywords: condensed matter physics, quantum integrable systems, spin chains, non-equilibrium physics, transport in one dimensional systems, asymptotic analysis, form factor approach.

The thesis is based on papers [1,2,3], where asymptotic properties of correlation functions of one dimensional quantum systems were considered. There are several reasons to study one-dimensional quantum physics.

First of all, quantum one-dimensional models have always attracted a lot of attention due to the rich mathematical structures, which appeared when their correlation functions were studied, and due to the possibility to address non-perturbative phenomena. The culmination of these developments resulted in the formulation of the Luttinger model which is an effective field theory at low temperatures.

Another reason is more practical. With the advancement of experimental techniques in cold atom experiments, it was observed a lot of unexpected properties in the behavior of 1d quantum systems. One of them is a remarkable result of *Kinoshita et al* (2006), where the authors observed for the system of cold atoms confined to move only in one dimension, that initially perturbed system did not approach an equilibrium state. Such behavior is typical for integrable systems, having a lot of conserved charges, which protect the system from thermalization. The interest in the non-equilibrium dynamics or dynamics of highly excited states motivated a lot of theoretical research resulting in new concepts such as generalized Gibbs ensembles, the quench action, generalized hydrodynamics, and others.

One more reason to study one dimensional quantum systems is due to miniaturization of electronic devices. It is believed that one day, leads may become effectively one dimensional. That is why, one should study transport in 1d systems.

One of the most remarkable phenomenon in this area is a quantization of conductance in quantum point contacts. Theoretical explanation of this phenomenon was done using Landauer–Büttiker formalism, which relates transmission matrix with conductance. Even though, according to the elementary theory of tunneling, the transmission probability is defined in a stationary setup, there was a lot of attention related to the non-equilibrium approach to the transport. The powerful analytic approaches involving Keldysh Green's function techniques were developed by different authors. From the point of view of the one-dimensional integrable models, the attention to similar problems was renewed in the context of the quantum quenches, which are specifically, understood as the evolution of the isolated quantum system initialized in the highly non-equilibrium state created either via the rapid change of the Hamiltonian or containing macroscopic spatial inhomogeneities.

As for the object of study, we mainly focus on correlation functions. The main approach to the correlation function in integrable models is a direct summation of the form-factors in the spectral expansion. The computation of the correlation functions at finite temperature, or more generally at finite entropy (density of states) is very different from the vacuum case due to the different decay rate of the form-factors with the system size (exponential vs power-law). Therefore, different approaches were developed to tackle this kind of problems including Quantum Transfer Matrix approach, non-linear differential equations, mainly based on the Riemann–Hilbert problem approach, the axiomatic definition of the thermal form-factors in the Integrable Quantum Field Theories, as well as partial summations of the few particle-hole excitations and extracting the most singular parts of the form-factors.

The thesis is based on three papers. In [1] the quantum XY spin chain was considered. The model was introduced in *Lieb et al* (1961) as a particular case of the Heisenberg XYZ spin chain model ($J_x \neq J_y \neq 0, J_z = 0$). The main reference for introducing into the subject is *Izergin et al* (2000), where authors derived a presentation for correlation function $G(m)$ of two spin operators (m is a distance between them) in terms of a difference of two Fredholm determinants. Based on this result we developed a method for asymptotic analysis of the correlation function for large distance m. We introduce effective form factors for one-dimensional lattice fermions with arbitrary phase shifts. We study tau functions defined as series of these form factors. On the one hand we perform the exact summation and present tau functions as Fredholm determinants in the thermodynamic limit. On the other hand simple expressions of form factors allow us to present the corresponding series as integrals of elementary functions. Using this approach we re-derive the asymptotics of static correlation functions of the XY quantum chain at finite temperature.

In [2], we considered the one-dimensional impenetrable lattice anyons. The model was introduced by *Paţu* (2015) as a generalization of the XXO quantum spin chain model. The author found two point correlation function $G(x,t)$ in the thermodynamic limit in terms of a difference of two Fredholm determinants. To describe large time and long distance behavior of these objects we used the effective form factor approach. In our study, we considered large distance x and large time t limit in such a way that its ratio remained a constant $x/t = \text{const}$. The asymptotic behavior is different in the space-like ($x/t > 1$) and time-like ($x/t < 1$) regions. In particular, in the time-like region we observed the additional power factor on top of the exponential decay. We argued that this result is universal as it is related to the discontinuous behavior of the phase shift function of the effective fermions. At particular values of the anyonic parameter ($\kappa = 1$), we recover asymptotics of spin-spin correlation functions in XXO quantum chain.

In [3], we studied transport in the free fermionic one-dimensional systems subjected to arbitrary local potentials. We considered a system consisting of two parts. The bias needed for the transport is modeled by the initial highly non-equilibrium distribution where only half of the system is populated. Additionally to that, the local potential is also suddenly changed when the transport starts. For such a quench protocol we computed the Full Counting Statistics (FCS) of the number of particles in the initially empty part. In the thermodynamic limit, the FCS was expressed via the Fredholm determinant with the kernel depending on the scattering data and Jost solutions of the pre-quench and the post-quench potentials. We studied the large-time asymptotic behavior of the obtained determinant and observed that if two or more bound states are present in the spectrum of the post-quench potential the information about the initial state manifests itself in the persistent oscillations of the FCS. On the contrary, when there are no bound states, the asymptotic behavior of the FCS is determined solely by the scattering data of the post-quench potential, which for the current (the first moment) is given by the Landauer–Büttiker formalism.

List of publications:

1. O. Gamayun, N. Iorgov and Y. Zhuravlev, *Effective free-fermionic form factors and the XY spin chain*, SciPost Physics **10**(3) (2021), doi: 10.21468/scipostphys.10.3.070 .

2. Y. Zhuravlev, E. Naichuk, N. Iorgov and O. Gamayun, *Large-time and long-distance asymptotics of the thermal correlators of the impenetrable anyonic lattice gas*, Physical Review B **105**(8) (2022), doi: 10.1103/physrevb.105.085145 .

3. O. Gamayun, Y. Zhuravlev and N. Iorgov, *On Landauer–Büttiker formalism from a quantum quench*, Journal of Physics A: Mathematical and Theoretical **56**(20), 205203 (2023), doi:10.1088/1751-8121/accabf .

Contents

Introduction

Actuality of theme.

This thesis is based on the papers [1, 2, 3] and devoted to the study of correlation functions in one dimensional quantum integrable systems. Below, we present a few key directions in this area and also review existing methods for computation and asymptotic analysis of correlation functions.

There are several reasons to study quantum one dimensional systems. One of them originates from the richness of mathematical structures of quantum integrable systems. The exact solvability of such systems allows to study non-perturbative effects. Let we say more about methods used in this area. The main subject of the study is correlation functions. Their computations can be divided into several steps. First of all, we mention the algebraic and coordinate Bethe ansatz which allow one to find spectrum and wave functions [4] and analytically address the thermodynamic properties of these systems [5]. The second step is usually devoted to the computation of the matrix elements of physical operators. They can be found analytically in many cases [6, 7, 8, 9], but the computation of correlation functions, which is presented as a form factor series, still remains quite challenging. For the vacuum correlation functions there are effective numerical methods based on integrability [10]. The asymptotic behaviour of the correlation functions can be investigated by means of effective field theory (Luttinger liquid) [11]. The origin of this behavior has been linked to the finite-size scaling of the matrix elements computed by means of the Bethe Ansatz [12, 13, 14, 15, 16]. For dynamical correlation functions based on this approach the corresponding effective field theory bears the name of *non-linear* Luttinger liquid [17, 18, 19]. At finite temperature, or more generally at finite entropy (density of states), both the numerical and field theory approaches experience some difficulties. In the numerical approaches one has to scan a much larger portion of Hilbert space to saturate the sum rules, as the form-factors (matrix elements of the physical operators) decay exponentially with the systems size contrary to the power-law decays at zero temperatures (see for instance [20, 21]). The field theory approach is based mainly on the linear spectrum for the soft modes (low-energy excitations) which is valid only for very low temperatures [22, 23]. A more rigorous approach was developed to evaluate finite temperature correlation function in integrable lattice models of Yang–Baxter type, based on the Quantum Transfer Matrix (QTM) [24]. The notion of the thermal form factor was introduced [25], which turned out to be useful for the asymptotic analysis of two-point correlation functions [25, 26, 27]. In the scaling limit, thermal form factors also arise axiomatically in the context of Integrable Quantum Field Theory [28, 29, 30, 31, 32, 33, 34, 35, 36]. Less rigorous but numerically accurate approaches are based on the thermodynamic limit of the form-factors and restricting summation to a finite number of particle-hole pairs [37, 38, 39, 40].

Another reason to study one dimensional quantum systems is motivated by the development of experimental techniques, where usually a system somehow becomes confined to move only in one dimension. For instance, in cold atom experiments [41, 42, 43]. It is hard not to mention the work [44], where authors observed that the considered system of cold atoms initially disturbed out of minimum did not tend to the equilibrium state even after a very long time. Such experiments boosted the interest in the non-equilibrium dynamics or dynamics of highly excited states and motivated a lot of theoretical research resulting in new concepts such as generalized Gibbs ensembles, the quench action [45, 21], generalized hydrodynamics (GHD) [46, 47, 48], and others. Another big area of experiments deals with the transport in one dimensional quantum systems. There is a remarkable phenomenon – quantization of the conductance [49] of the electric current through the quantum point contact. The theory which explains such phenomena called Landauer–Büttiker formalism [50, 51, 52]. It directly allows one to express the conductance in terms of the transmission matrix, this way relating transport and quantum properties [53, 54]. From the point of view of the one-dimensional integrable models the attention to transport problems was renewed in the context of the quantum quenches, which are specifically, understood as the evolution of the isolated quantum system initialized in the highly non-equilibrium state created either via the rapid change of the Hamiltonian or containing macroscopic spatial inhomogeneities [55, 56, 57, 58, 59]. The latter is more pertinent to the quantum transport setup and is dubbed as the partition approach [60, 61]. The large-time behavior of such systems can be described by the generalized hydrodynamics [62, 46], which allows one to get analytic treatment of the non-equilibrium steady currents, describe anomalous diffusion, and address the correlation functions (for review see the special issue [63]).

The dissertation is devoted to the study of correlation functions in one dimensional quantum systems. More precisely, it focuses on derivation of exact expressions for correlation functions and asymptotic analysis of them in the thermodynamic limit (size of the system goes to infinity). We develop a new method called the effective form factor approach, which allow one to study asymptotics of the correlation functions presented in terms of Fredholm determinants.

Scientific programs, plans, topics, grants related to the dissertation.

The dissertation work was carried out in the Department of Mathematical Methods in Theoretical Physics of Bogolyubov Institute for Theoretical Physics of the National Academy of Sciences of Ukraine within the framework of the academic theme "Symmetries, deformations and integrability in models of quantum fields and particles", state registration number in UkrINTEI 0117U00023 (2017 - 2021) and also "Deformation and symmetry aspects and integrability and exact solvability of quantum physics models", state registration number in UkrINTEI 0122U000888 (2022-2026).

In addition, the works included in the thesis were partially supported by the National Research Foundation of Ukraine grant 2020.02/0296 "Equilibrium and non-equilibrium processes in quantum integrable models in condensed matter physics" (2020 - 2021, 2023).

The purpose and objectives of the research.

The objectives of this thesis address equilibrium and non-equilibrium processes in one dimensional quantum systems. The main peculiarity of these systems is that the standard methods usually cannot be applied there. For instance, the Landau Fermi-liquid theory that describes fermions in 3D is not applicable for one-dimensional quantum systems. Instead, one has to introduce Tomonaga–Luttinger theory, along with new mathematical methods such as bosonization and soft-modes summation. The applicability of the Tomonaga-0Luttinger theory and its various generalizations to the description of the specific physical one-dimensional systems is of interest to many groups in the world and presents the main direction of the thesis. The thesis is focused mainly on quantum integrable systems. Integrability allows us to advance in the exact computations and address nonperturbative regimes. Even when it is impossible to perform analytic calculations to the very end, the intermediate results significantly facilitate numerical calculations and asymptotic analysis. The exact computations require application of the modern algebraic and analytical mathematical methods: two-dimensional conformal field theory, the nonlinear steepest descent method in the matrix Riemann–Hilbert problem, combinatorial methods of the form factor summations etc. The necessity to apply these mathematical methods to specific physical problems naturally brings yet another goal of this thesis — the improvement and expansion of the methods themselves. The obtained results are expected to be important both for physical applications and pure mathematics.

Object of the research is equilibrium and non-equilibrium processes in one dimensional quantum integrable systems.

Subject of the research is correlation functions and its asymptotics for the one dimensional quantum integrable systems, which allows one to study static and dynamical properties of the system.

Methods of the research.

We use a synthesis of various mathematical methods. In particular, we employ methods for summing form-factors in quantum models associated with free fermions (XY model, impenetrable bosons, etc.). The result of such summations consists in the presentation of the correlation functions as Fredholm determinants. Our main method to find the asymptotic behavior of the correlation functions consists in the direct extraction of it from the form-factor series in the finite system. However, the number of summands in the form-factor series for correlation functions at finite temperature or in the non-equilibrium setup is increased exponentially with the system size. Therefore, in this thesis, we develop an alternative method that modifies form factors but reduces the sums to the vacuum case, this way simplifying the form factor summation. Finally, to check predictions for correlation function in integrable models we use numerical methods based on [64], which allow one to compute Fredholm determinants directly via discretization procedure.

Scientific novelty of the obtained results.

The following original results were obtained in the dissertation:

- The method of effective form factors was introduced and applied to the study of asymptotics of correlation functions in different quantum one dimensional models.

- In particular, it was re-derived asymptotics for correlation functions in the XY model and more general asymptotics of Toeplitz determinants with continuous symbol having arbitrary integer winding number.

- Additionally, it was obtained asymptotics of the correlation functions in the anyonic model for large time and distance in time-like and space-like regimes. In the time-like regime, the effective phase $\nu(p)$ becomes discontinuous. We developed regularization procedure and obtained additional power-law prefactor on top of the exponential decay.

- We derived the Full Counting Statistics for the 1d transport via an arbitrary defect from the first principles. The answer was expressed via the Fredholm determinant. Large-time asymptotic behavior of the obtained Fredholm determinant was reduced to the determinant of the sine-kernel type. The answer depends only on the transmission coefficient of the post-quench potential, while traces of the original state are present only as the energy distribution. When there are two or more bound states present in the spectrum of the post-quench potential, the FCS gets persistent oscillating behavior.

Personal contribution of the PhD candidate.

In work [1], it was found finite-size scaling of the effective form factors for negative values of the winding number δ, having the form very similar to the work of [65]. It was obtained the relation between two representations of $G(m)$, the first one as the difference of two Fredholm determinants and the second one as the single Fredholm determinants. Obtained analytical formulas for asymptotics of correlation functions $G(m)$ was presented in a convenient way for comparing with existing results [66, 65]. For values of the winding numbers $\delta > 1$, it was performed a summation of the tau function in the Fredholm determinant.

Using discretization methods for computing Fredholm determinants [64], it was performed numerical checks for the obtained analytical formulas for correlation function $G(m)$ for different values of the winding number $\delta = 0, \pm 1$.

In work [2], it was rewritten the answer obtained in [67] for the correlation function in the anyonic model in a way convenient for using the effective form factor approach. It was observed that solutions of the equation for phase shift $\nu(p)$ may have discontinuities for large time and distance in the time-like regime. It was developed a regularization procedure for singular solutions of the equation for phase shift $\nu(p)$, which led to the additional power law behavior of the correlation function. Using generalization of the steepest descent method, namely Watson's lemma, the integral over holes was performed in the time-like regime. The obtained formula was compared with [68], where a similar result was obtained for the correlation function of the XX model. Numerical checks was performed for the obtained analytical formulas for correlation function $G(x, t)$ in space-like and time-like regimes.

In work [3], it was observed that introducing Green's function $G(x, y, t)$ drastically simplifies taking thermodynamic limit for the full counting statistics. For comparison, one can read Appendices F and G, where the similar computations is done but without introducing the Green's function. The case of two delta potential barrier was analyzed, where by properly setting parameters two bounded states may appear. In this case, it was observed that the current through the barrier and the full counting statistics oscillate with the frequency proportional to the difference of the energies of the bounded states. Numerical checks was done for the full counting statistics, showing that it can be approximated by the Fredholm determinant with more simple kernel, namely sine kernel.

Practical significance of the obtained results. The results obtained in the dissertation can be used in the study of dynamical properties of quantum systems as an alternative to the existing methods such as the Riemann-Hilbert approach, thermodynamic Bethe ansatz, the Luttinger liquid theory etc. The main advantage of the effective form factor approach is its relative simplicity compared to the existing methods.

Approbation of the results of the dissertation. The results highlighted in the dissertation were presented at the seminars of Bogolyubov Institute for

Theoretical Physics of the NAS of Ukraine, as well as on international conferences held in Kyiv, see Appendix J.

Publications. The results of this dissertation are presented in 3 journal publications [1, 2, 3], see also Appendix I.

Structure of the dissertation. The dissertation consists of an introduction, 3 chapters corresponding to logically completed stages of research, conclusions, bibliography, which contains 182 references and 10 appendices. The dissertation includes 10 figures. The total volume of work is 159 pages of printed text.

Chapter 1

Effective free-fermionic form factors and the XY spin chain

1.1 Introduction

This chapter is based on [1], where we developed a heuristic approach to address correlation functions at finite density of states. Our main motivating example is the XY spin chain in a transverse magnetic field. On one hand it allows to get exact answers for spin-spin correlation functions in terms of Toeplitz or Fredholm determinants and on the other hand matches the complexity of generic systems. As we have mentioned above, this complexity is combinatorial in nature and reflects the fact that each form factor for the thermal states is exponentially small so the number of relevant form factors is exponentially large. This makes direct computation of the corresponding sum for the correlation functions notoriously difficult and force researchers to focus at most on the two particle-hole excitations [37, 38, 39, 40], consider semiclassical approximations [69] or develop other approximation schemes [70, 71].

We deal with this problem in a different manner. Namely, to describe the spin-spin correlation function evaluated on a state with finite density of entropy (energy) we introduce *effective* form factors for the fermions with the modified phase shift that absorbs information about the state and significantly simplifies combinatorics of excitations making it essentially analogous to the zero-temperature case. Here we have to emphasize that the expressions of form factors was inspired by the XX spin chain [72, 73, 74, 75], rather than genuine spin form factors in the Ising/XY models [76, 77, 78, 79, 80].

We focus on the static correlation functions for which we demonstrate that after complete summation of the effective form factors series and taking the thermodynamic limit the answer can be presented in the form of Fredholm determinants.

For the proper choice of the phase shift the kernels in the Fredholm determinants differ from the exact ones [75] by the exponentially small (in distance between spin operators) terms. Conversely, by first taking the thermodynamic limit of the effective form factors and then performing their summation we manage to present the Fredholm determinants as integrals of elementary functions. This kind of asymptotic behavior for models in the continuum (not the lattice) arises similarly from the solution of the Riemann–Hilbert problem for operators acting on the whole real line [81]. This asymptotics was conjectured to be universal for correlation functions of any gapless model of statistical mechanics at any temperature and for an arbitrary coupling constant [82].

An important ingredient for our asymptotic analysis is the winding number of the state-dependent phase shift $\nu(q)$ defined as the difference across the Brillouin zone, namely

$$\nu(+\pi) - \nu(-\pi) = \delta \in \mathbb{Z}. \tag{1.1}$$

We recover the correlation length in the lattice version of the asymptotics in Ref. [82] at $\delta = 1$ and additionally give an analytic expression for the prefactor. For $\delta = 0$ and $\delta = \pm 1$ we derive asymptotic behavior for the correlation function in the XY spin chain at finite temperature and compare it with the known answer [66]. Different winding numbers correspond to different values of the magnetic field and anisotropy. The winding number $|\delta| \geq 2$ does not have a direct physical interpretation in this model, but we perform the asymptotic analysis anyway and find the results consistent with the generalization of Szegő formulas [65]. Moreover, we have observed a peculiar identity between Toeplitz determinants and Fredholm determinant of sine-kernel type with finite rank, which, to the best of our knowledge, is new

$$\det\left(1 + \hat{S}_\nu + \delta\hat{V}_\nu\right) - \det\left(1 + \hat{S}_\nu\right) = \det_{1\leq j,k\leq x} T_{j-k}, \tag{1.2}$$

where the operators $\hat{S}_\nu$ and $\delta\hat{V}_\nu$ are generalized sine-kernels that act on $L^2([-\pi,\pi])$ and are defined by their kernels

$$S_\nu(p,q) = \frac{e^{2\pi i\nu(p)} - 1}{2\pi} e^{i(p-q)/2} \frac{\sin\frac{x(p-q)}{2}}{\sin\frac{p-q}{2}}, \tag{1.3}$$

$$\delta V_\nu(p,q) = -\frac{e^{2\pi i\nu(p)} - 1}{2\pi} e^{-ix(p+q)/2}, \tag{1.4}$$

and

$$T_k = -\frac{1}{2\pi} \int_{-\pi}^{\pi} d\varphi \, e^{-i(k-1)\varphi + 2\pi i \nu(\varphi)}. \tag{1.5}$$

Notice that the right hand side of this formula can be also be presented as a Fredholm determinant but with modified shifted $\nu(k)$ [83]. For $\nu(q)$ corresponding to the XY spin chain Eq. (1.2) gives different representations of the spin-spin correlation function, the left hand side was obtained in [75] and the right hand side in [84, 66].

The chapter is organized as follows. In Sec. 1.2 we define the tau function together with the effective form factors and outline the derivation of the Fredholm determinant presentation resulting from the summation of form factors. The details of this derivation are presented in Appendix A. In Sec. 1.3 we study the thermodynamic limit of the form factors and argue for an explicit presentation of the form factors series as integrals of elementary functions. All necessary technical results are given in Appendices B and C. Sec. 1.4 deals with the application of the general formulas to the XY spin chain. In Sec. 1.5 we discuss connection of the general result to the Toeplitz determinant and relations such as Eq. (1.2). Sec. 1.6 concludes the chapter and offers an outlook.

1.2 Effective form factors

We start with the formal definition of the static correlation function (tau function), as a form-factor series

$$\tau(x,t) \equiv \sum_{\mathbf{q}} |\langle \mathbf{k}|\mathbf{q}\rangle|^2 e^{-ix(\sum\limits_{i=1}^{N+1} k_i - \sum\limits_{i=1}^{N} q_i)}, \tag{1.6}$$

here the ordered set $\mathbf{k} = \{k_1, \dots k_{N+1}\}$ consists of $N+1$ distinct shifted momenta inside the Brillouin zone ($k_i \sim k_i + 2\pi n$, $n \in \mathbb{Z}$) each being a solution of the transcendental equation

$$e^{ikL} = e^{-2\pi i \nu(k)} \tag{1.7}$$

for a smooth function $\nu(k)$. This function plays the role of the phase shift and is assumed to be compatible with the Brillouin zone structure, i.e.

$$\nu(\pi) - \nu(-\pi) = \delta \in \mathbb{Z}. \tag{1.8}$$

The integer δ is the winding number (index). One can easily argue that the number of solutions of Eq. (1.7) is $L+\delta$, each root defined up to $O(1/L^2)$ terms.

The set $\mathbf{q} = \{q_1, \ldots, q_N\}$ is an ordered set of N distinct solutions of equation

$$e^{iqL} = 1. \tag{1.9}$$

Motivated by the spin form factors for quantum XY chain written in the XXO basis [75], we postulate the following form-factor

$$|\langle \mathbf{k}|\mathbf{q}\rangle|^2 = -\frac{4L}{\prod\limits_{i=1}^{N+1}(1+\frac{2\pi}{L}\nu'(k_i))}\left(\prod_{i=1}^{N+1}\frac{e^{g(k_i)/2}\sin\pi\nu_i}{L}\right)^2\prod_{i=1}^{N}e^{-g(q_i)}(\det D)^2, \tag{1.10}$$

where $\det D$ is a trigonometric Cauchy type determinant that can be presented in two equivalent forms

$$\det D = \begin{vmatrix} \cot\frac{k_1-q_1}{2} & \ldots & \cot\frac{k_{N+1}-q_1}{2} \\ \vdots & \ddots & \vdots \\ \cot\frac{k_1-q_N}{2} & \ldots & \cot\frac{k_{N+1}-q_N}{2} \\ 1 & \ldots & 1 \end{vmatrix} = \frac{\prod\limits_{i>j}^{N+1}\sin\frac{k_i-k_j}{2}\prod\limits_{i>j}^{N}\sin\frac{q_j-q_i}{2}}{\prod\limits_{i=1}^{N+1}\prod\limits_{j=1}^{N}\sin\frac{k_i-q_j}{2}}. \tag{1.11}$$

Furthermore, since we do not specify the specific operator we will sometimes refer to Eq. (1.10) as to the overlap, and use this term interchangeably with form factor.

We assume that the index is of order $\delta \sim O(1)$ as both the system size and the number of particles are approaching the thermodynamic limit $N\to\infty$, $L\to\infty$ such that $N/L = 1$. In this case, the summation over $\mathbf{q}$ can be performed exactly (see Appendix A) and the result for the tau function reads

$$\tau(x) \overset{N\to\infty}{=} \det(1+\hat{V}+\delta\hat{V}) - \det(1+\hat{V}), \tag{1.12}$$

where the determinants are taken in the space $L^2(S^1)$ and the corresponding operators are defined by their kernels

$$V(k,q) = \frac{\sin^2(\pi\nu(k))}{4\pi}e^{g(k)}e^{-i(k+q)x/2}e^{i(k-q)/2}\frac{E(k)-E(q)}{\sin\frac{k-q}{2}}, \tag{1.13}$$

$$\delta V(k,q) = -\frac{2}{\pi}\sin^2(\pi\nu(k))e^{g(k)}e^{-i(k+q)x/2}, \qquad k,q\in[-\pi,\pi) \tag{1.14}$$

with

$$E(k) = \int_{-\pi}^{\pi} \frac{dq}{\pi} e^{-g(q)+iqx} \cot \frac{q+i0-k}{2} - \frac{4ie^{-g(k)+ikx}}{e^{-2\pi i\nu(k)}-1}. \tag{1.15}$$

The diagonal terms $k = q$ are understood as in L'Hopital's limiting procedure. Further, we impose the relation

$$e^{-g(k)} = e^{-2\pi i\nu(k)} - 1. \tag{1.16}$$

to present tau function as

$$\tau(x) = \det(1 + \hat{S}_\nu + \delta\hat{V}_\nu + \hat{R}) - \det(1 + \hat{S}_\nu + \hat{R}) \tag{1.17}$$

with $\hat{S}_\nu$ being a generalized sine-kernel

$$S_\nu(k,q) = \frac{e^{2\pi i\nu(k)}-1}{2\pi} e^{i(k-q)/2} \frac{\sin\frac{x(k-q)}{2}}{\sin\frac{k-q}{2}}, \tag{1.18}$$

$$\delta V(k,q) = -\frac{e^{2\pi i\nu(k)}-1}{2\pi} e^{-ix(k+q)/2}. \tag{1.19}$$

This way, the remainder $\hat{R} = \hat{V} - \hat{S}_\nu$ consists of integrals in Eq. (1.15), which are exponentially suppressed[1] for large and positive x. Let us call the tau function with discarded $\hat{R}$ as τ_S, namely

$$\tau_S(x) = \det(1 + \hat{S}_\nu + \delta\hat{V}_\nu) - \det(1 + \hat{S}_\nu). \tag{1.20}$$

This particular generalization of the sine-kernel is contained in the prefactor $(e^{2\pi i\nu(k)} - 1)$ and allows one to describe a modification of the system from the vacuum state for which $\nu(q)$ is constant within the arc $k \in [-k_F, k_F]$ and zero everywhere else, to the finite-entropy state, where, for instance, for the thermal state of the fermionic system the prefactor would be proportional to the single-particle Fermi distribution function[2]. In Sec. 1.4 we relate this type of kernel to the static spin-spin correlations in the XY chain. Then $\nu(k)$ will depend not only on the state but also on the parameters of the model.

[1] We assume that $\exp(-2\pi i\nu(q))$ is an analytic function within some vicinity of the real line.

[2] See, for instance the discussion in Appendix A in Ref. [85].

1.3 Thermodynamic limit and direct summation of form factors

1.3.1 Winding number $\delta = 1$

In the previous section, we considered the summation of the form factor series and subsequent taking of the thermodynamic limit. This leads to the presentation of the tau function as a Fredholm determinant. The essence of this derivation, which is outlined in Appendix A, is that each momentum $q_i \in \mathbf{q}$ was treated independently. In this section, we focus more on the detailed structure of the ordered sets $\mathbf{q}$ in the sum of Eq. (1.6). The total number of solutions of the equation $e^{iqL} = 1$ inside the Brillouin zone is L, which can be presented as

$$q_j = \frac{2\pi}{L}\left(-\frac{L+1}{2} + j\right), \qquad j = 1, 2, \ldots, L. \tag{1.21}$$

As we have already pointed out above, the number of solution of Eq. (1.7), depends on the winding number δ. In particular, for $\delta = 1$ there exist exactly $L + 1$ solutions inside the Brillouin zone

$$k_j = \frac{2\pi}{L}\left(-\frac{L+1}{2} + j - \nu_j\right), \quad \nu_j = \nu(k_j) \approx \nu(q_j), \quad j = 1, \ldots, L+1. \tag{1.22}$$

If we choose the set $\mathbf{k} = \{k_1 \ldots, k_{L+1}\}$ in Eq. (1.6) then summation over $\mathbf{q}$ will only involve one term $\mathbf{q} = \{q_1, \ldots q_L\}$. In the large L limit the corresponding overlap reduces to a constant which is slightly counterintuitive from the orthogonality catastrophe point of view [86]. The explicit value of this constant is given by Eq. (C.11). The difference of momenta in Eq. (1.6) can be evaluated in the large L limit as

$$\Delta P \equiv \sum_{i=1}^{L+1} k_i - \sum_{i=1}^{L} q_i \approx \pi - \int_{-\pi}^{\pi} \nu(q) dq. \tag{1.23}$$

Combining these observations together we obtain explicit equality for the Fredholm determinants in Eq. (1.12), and approximation for the generalized sine-kernel

$$\tau_S(x) \approx \tau(x) = \exp\left(-i\pi x + ix\int\limits_{-\pi}^{\pi}\nu(q)dq\right)$$
$$\times \exp\left(-\frac{1}{2}\int\limits_{-\pi}^{\pi}dq\int\limits_{-\pi}^{\pi}dk\left[\frac{\nu(q)-\nu(k)-(q-k)/2\pi}{2\sin\frac{q-k}{2}}\right]^2\right). \quad (1.24)$$

It is interesting to note that only the periodic (i.e. having winding number $\delta = 0$) part of $\nu(q)$ has entered the final answer.

1.3.2 Winding number $\delta = 0$

For $\delta = 0$ we proceed similarly to the previous subsection. This time however the maximal possible number of the $k_i \in \mathbf{k}$ is L, so the maximal set $\mathbf{q}$ consists of $N = L-1$ momenta. There are exactly L such sets and they can be parameterized by the position of the "hole"

$$\mathbf{q}^{(a)} = \{q_1, \ldots, q_{a-1}, q_{a+1}, \ldots, q_L\}, \qquad a = 1, 2, \ldots, L. \quad (1.25)$$

The overlap is given by

$$|\langle \mathbf{k}|\mathbf{q}^{(a)}\rangle|^2 = \frac{A[q_a]e^{g(q_a)}}{L}\left[\frac{\Gamma(L-a+1-\nu_a)\Gamma(a+\nu_a)}{\Gamma(L-a+1-\nu_+)\Gamma(a+\nu_+)}\right]^2\left(\frac{\pi+q_a}{\pi-q_a}\right)^{2\nu_+-2\nu_a}. \quad (1.26)$$

The derivation can be found in Appendix C.3 along with the explicit expression for $A[q_a]$ (see Eq. (C.25)). For $a \sim L$ and $L - a \sim L$ the last two factors cancel each other, which yields the following explicit expression

$$|\langle \mathbf{k}|\mathbf{q}^{(a)}\rangle|^2 = \frac{e^{-2\pi i\nu_a}-1}{L}$$
$$\times \exp\left(-\frac{1}{2}\int\limits_{-\pi}^{\pi}dq\int\limits_{-\pi}^{\pi}dk\left[\frac{\nu(q)-\nu(k)}{2\sin\frac{q-k}{2}}\right]^2 - \int\limits_{-\pi}^{\pi}\nu(q)\cot\frac{q-q_a+i0}{2}dq\right). \quad (1.27)$$

On a technical side, we have used a variation of Sokhotski–Plemelj formula

$$⨍_{-\pi}^{\pi}\nu(q)\cot\frac{q-k}{2}dq = \int_{-\pi}^{\pi}\nu(q)\cot\frac{q-k+i0}{2}dq + 2\pi i\nu(k), \quad (1.28)$$

to transform the integral in the exponential. For $a \sim 1$ and $L-a \sim 1$, the overlap is still $O(1/L)$, so we can replace the sum in tau function Eq. (1.6) by an integral

$$\tau(x) = e^{-ix\sum\limits_{j=1}^{L}(k_j-q_j)} \sum_{a=1}^{L} |\langle \mathbf{k}|\mathbf{q}^{(a)}\rangle|^2 e^{-ixq_a} = T_0(x)Y_0(x) \tag{1.29}$$

with

$$T_0(x) = \exp\left(ix\int\limits_{-\pi}^{\pi} \nu(q)dq - \frac{1}{2}\int\limits_{-\pi}^{\pi} dq \int\limits_{-\pi}^{\pi} dk \left[\frac{\nu(q)-\nu(k)}{2\sin\frac{q-k}{2}}\right]^2 \right), \tag{1.30}$$

and

$$Y_0(x) = \int\limits_{-\pi}^{\pi} \frac{dk}{2\pi}(e^{-2\pi i\nu(k)}-1)e^{-ikx} \exp\left(-\int\limits_{-\pi}^{\pi} \nu(q)\cot\frac{q-k+i0}{2}dq \right). \tag{1.31}$$

Equivalently we may re-write $Y_0(x)$ as a contour integral in the variable $z = e^{ik}$

$$Y_0(x) = \frac{1}{2\pi i}\oint_{C_>} \frac{dz}{z}(e^{-2\pi i\nu(k)}-1)z^{-x}\mathfrak{S}(z), \tag{1.32}$$

where the contour $C_>$ is a circle centered at the origin with slightly larger than unit radius and

$$\mathfrak{S}(z) = \exp\left(i\int\limits_{-\pi}^{\pi} dq\, \nu(q)\frac{z+e^{iq}}{z-e^{iq}} \right). \tag{1.33}$$

We assume that $\nu(q)$ is non-singular in the region of integration, thus $\mathfrak{S}(z)$ is holomorphic outside the unit circle on the Riemann sphere, so the asymptotic for large positive integers x is defined by the analytic behavior of $\nu(k)$ in the upper-half plane. For example, if $e^{-2\pi i\nu(k)}$ is a meromorphic function (of $z = e^{ik}$) outside the unit circle in the complex plane having simple poles at z_1, $z_2, \ldots$, with $1 < |z_1| < |z_2| < \cdots$, then for large x the leading contribution comes from the smallest pole

$$Y_0(x) \approx -z_1^{-x-1}\mathfrak{S}(z_1)\,\mathrm{res}_{z=z_1}\, e^{-2\pi i\nu(k)}, \qquad z = e^{ik}. \tag{1.34}$$

Applying this formula together with Eq. (1.30), we have an asymptotic expression for the sine-kernel Fredholm determinant for $\delta = 0$

$$\tau_S(x) \approx \tau(x) \approx -\frac{T_0(x)}{z_1^{x+1}}\mathfrak{S}(z_1)\,\mathrm{res}_{z=z_1}\, e^{-2\pi i\nu(k)}. \tag{1.35}$$

Remark. Notice that even for $\delta = 1$ one could have chosen $\mathbf{k} = \{k_1, \dots k_L\}$. This would not affect the derivation of the Fredholm determinants, but instead of one term in the form factor series as in the previous section, we still get a sum of L terms. Using Appendix (C.3) and specifically Eq. (C.27), we obtain

$$\tau(x) = e^{-ix\sum_{j=1}^{L}(k_j - q_j)} \sum_{a=1}^{L} |\langle \mathbf{k}|\mathbf{q}^{(a)}\rangle|^2 e^{-ixq_a} = \tau_{\delta=1}(x)$$
$$\times \sum_{a=1}^{L} e^{ix\pi - ixq_a} \frac{\sin^2(\pi\nu_a)}{\pi^2} \frac{e^{2F(q_a)+g(q_a)}}{e^{2F(\pi)+g(\pi)}} \left[\frac{\Gamma(L-a+1-\nu_a)\Gamma(a+\nu_a)}{\Gamma(L-a+2-\nu_+)\Gamma(a+\nu_+)}\right]^2. \quad (1.36)$$

Here, by $\tau_{\delta=1}(x)$ we mean the r.h.s of Eq. (1.24). Notice that contrary to the $\delta = 0$ scenario, the middle parts $a \sim L$ and $L-a \sim L$, are suppressed as $1/L^2$, so their contributions are negligible as $L \to \infty$. The soft-modes at the edges $a \ll L$ and $L - a \ll L$ now start to play more important role because the corresponding overlaps are $O(1)$. The prefactor in front of the Gamma functions simplifies to one and the whole series reads

$$\tau(x) = \tau_{\delta=1}(x) \frac{\sin^2(\pi\nu_-)}{\pi^2} \sum_{a=1}^{L} \left[\frac{\Gamma(L-a+1-\nu_a)\Gamma(a+\nu_a)}{\Gamma(L-a+2-\nu_+)\Gamma(a+\nu_+)}\right]^2 + O(1/L). \quad (1.37)$$

In order to compute this sum in the $L \to \infty$ limit we expand it at the edges and then perform the summation of the simplified expression extending the upper limit to infinity. Namely, the asymptotics

$$\frac{\Gamma(L-a+1-\nu_a)\Gamma(a+\nu_a)}{\Gamma(L-a+2-\nu_+)\Gamma(a+\nu_+)} = \begin{cases} (a+\nu_-)^{-2} & , a \ll L \\ (L-a-\nu_-)^{-2} & , L-a \ll L \end{cases}, \quad (1.38)$$

leads to

$$\frac{\tau(x)}{\tau_{\delta=1}(x)} = \frac{\sin^2(\pi\nu_-)}{\pi^2} \left(\sum_{a=1}^{\infty} \frac{1}{(a+\nu_-)^2} + \sum_{a=0}^{\infty} \frac{1}{(a-\nu_-)^2}\right)$$
$$= \frac{\sin^2(\pi\nu_-)}{\pi^2} \sum_{a=-\infty}^{\infty} \frac{1}{(a+\nu_-)^2} = 1. \quad (1.39)$$

This way we restore the correct result even in the different formulation of the form-factor series.

1.3.3 Negative winding number $\delta < 0$

Let us consider $\delta = 1 - n$ for positive integers $n \in \mathbb{Z}_{>}$. The maximal number of solutions of Eq. (1.7) is $\ell = L + \delta$. We choose all of them to comprise our set $\mathbf{k}$

$$\mathbf{k} = \{k_1, \dots k_\ell\}, \qquad k_i = \frac{2\pi}{L}\left(-\frac{L+1}{2} + i - \nu_i\right). \tag{1.40}$$

The set $\mathbf{q}^{a_1,\dots a_n}$ is obtained from the complete set $\mathbf{q}$ in Eq. (1.21) by the omission of the "particle" (creating a "hole") at positions q_{a_i}

$$\mathbf{q}^{a_1,\dots a_n} = \{q_1, \dots \hat{q}_{a_1}, \dots \hat{q}_{a_n}, \dots q_L\}. \tag{1.41}$$

The total difference of momenta for such a state reads

$$\Delta P_{a_1,\dots a_n} = \sum_{i=1}^{\ell} k_i - \sum_{i=1}^{L} q_i + \sum_{i=1}^{n} q_{a_i} \approx \delta\pi - \int\limits_{-\pi}^{\pi} \nu(q)dq + \sum_{i=1}^{n} q_{a_i}. \tag{1.42}$$

The corresponding overlap is analyzed thoroughly in Appendix C.4 for $a_i \sim L$, $L - a_i \sim L$. It gives the following contribution to the tau function (1.6)

$$e^{-ix\Delta P_{a_1,\dots a_n}} |\langle \mathbf{k}|\mathbf{q}^{a_1,\dots a_n}\rangle|^2 = \mathcal{A}_\delta[\nu] \prod_{i>j}^{n} \left(2\sin\frac{q_{a_i} - q_{a_j}}{2}\right)^2 \prod_{i=1}^{n} \mathcal{Y}_{a_i}, \tag{1.43}$$

where

$$\mathcal{A}_\delta[\nu] = \exp\left(ix\int\limits_{-\pi}^{\pi} \nu(q)dq - ix\delta\pi\right) \times \left(-\frac{1}{2}\int\limits_{-\pi}^{\pi} dq \int\limits_{-\pi}^{\pi} dk \left[\frac{\nu(q) - \nu(k) - (q-k)\delta/(2\pi)}{2\sin\frac{q-k}{2}}\right]^2\right) \tag{1.44}$$

and

$$\mathcal{Y}_a = -4\frac{\sin^2(\pi\nu(q_a))}{L} \times \exp\left[-ixq_a + g(q_a) - \fint\limits_{-\pi}^{\pi} dq \left(\nu(q) - \delta\frac{q}{2\pi}\right)\cot\frac{q-q_a}{2}\right]. \tag{1.45}$$

In order to evaluate the tau function we proceed similarly to Ref. [87, 88] (see also [89]). First, we notice that one can present the product of sines in (1.43) as Vandermonde determinants

$$\prod_{i>j}^{n}\left(2\sin\frac{q_{a_i}-q_{a_j}}{2}\right)^2=\prod_{k>j}^{n}\left(e^{iq_{a_k}}-e^{iq_{a_j}}\right)\left(e^{-iq_{a_k}}-e^{-iq_{a_j}}\right)$$
$$=\det_{1\le j,k\le n}\left(e^{i(j-1)q_{a_k}}\right)\det_{1\le j,k\le n}\left(e^{-i(j-1)q_{a_k}}\right)$$
$$=\varepsilon_{j_1\dots j_n}\varepsilon_{j'_1\dots j'_n}e^{i(j_1-j'_1)q_{a_1}}\dots e^{i(j_n-j'_n)q_{a_n}},\quad (1.46)$$

where $\varepsilon_{j_1\dots j_n}$ is a completely antisymmetric tensor; the summation over repeated indices is implied. This expression is an almost factorized, so in the second step we render summation over q_{a_i} to be independent, namely

$$\sum_{q_{a_1}<\dots<q_{q_n}}=\frac{1}{n!}\sum_{q_{a_1}}\dots\sum_{q_{a_n}}.\quad (1.47)$$

This immediately allows us to write the tau function (1.6) in the thermodynamic limit

$$\tau(x)=\det_{1\le j,k\le n}\left[Y_\delta(x+j-k)\right]\times$$
$$\exp\left(ix\int_{-\pi}^{\pi}\nu_\delta(q)dq-\frac{1}{2}\int_{-\pi}^{\pi}dq\int_{-\pi}^{\pi}dk\left[\frac{\nu_\delta(q)-\nu_\delta(k)}{2\sin\frac{q-k}{2}}\right]^2\right),\quad (1.48)$$

where $\nu_\delta(q)\equiv\nu(q)-(q+\pi)\delta/(2\pi)$ has zero winding number and $Y_\delta(x)$ stands for

$$Y_\delta(x)=\int_{-\pi}^{\pi}\frac{dq}{2\pi}\left(e^{-2\pi i\nu(q)}-1\right)\times$$
$$\exp\left(-i(x-\delta)q+i\delta\pi-\int_{-\pi}^{\pi}dk\nu_\delta(k)\cot\frac{q-k+i0}{2}\right).\quad (1.49)$$

The integral has been transformed using identities such as Eq. (1.28) to facilitate finding asymptotic behavior at large positive x. Indeed, the exponential is an analytic function, so after the proper deformation of the integration contour it can be dropped. This way, we demonstrate that, in fact, $Y_\delta(x)$ depends only on $\nu_\delta(x)$, namely

$$Y_\delta(x) = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{-2\pi i \nu_\delta(q)} \exp\left(-ixq - \int_{-\pi}^{\pi} dk \nu_\delta(k) \cot \frac{q-k+i0}{2} \right). \tag{1.50}$$

Let us emphasize that Eq. (1.48) gives the exact answer for the Fredholm determinants (1.12). The asymptotic behavior for large x will give also asymptotics for the generalized sine-kernel determinants (1.20). Similar to the treatment of the $\delta = 0$, the asymptotic expansion of $Y_\delta(x)$ is connected with analytic properties of $\nu(q)$. Let us assume that the first n leading terms are given by

$$Y_\delta(x) = A_1 e^{-\varkappa_1 x} + \cdots + A_n e^{-\varkappa_n x} + o(e^{-\varkappa_n x}), \qquad |\varkappa_i| \leq |\varkappa_{i+1}|. \tag{1.51}$$

Then the leading order of the determinant reads

$$\det_{1\leq j,k\leq n} [Y_\delta(x+j-k)] \approx \prod_{i=1}^{n} A_i e^{-\varkappa_i x} \prod_{i>j}^{n} \left(2\sinh \frac{\varkappa_i - \varkappa_j}{2} \right)^2 . \tag{1.52}$$

For $n = 1$ we reproduce the results from the previous subsection.

1.3.4 Positive winding numbers $\delta > 1$

For $\delta > 1$, similar to $\delta = 1$, we can keep the maximal available number of $k_i \in \mathbf{k}$, so the r.h.s sum in Eq. (1.6) consists of one term, which is of order $O(1/L)$. This means that the corresponding Fredholm determinants in Eq. (1.12) vanish identically. The reason for this can already be seen before going into the thermodynamic limit. Namely, first we notice that the matrix $\mathcal{A}_{ij}$ in Eq. (A.10) can be considered on the full set of momenta $\mathbf{k} = k_1, \dots k_{L+\delta}$, which will not change the determinant's limiting value

$$\lim_{L\to\infty} \det_{1\leq i,j\leq L+\delta} \mathcal{A}_{ij} = \det(1+\hat{V}). \tag{1.53}$$

But since $\mathcal{A}_{ij}$ has the form of Eq. (A.8) which can be schematically written as

$$\mathcal{A}_{ij} = \sum_{k=1}^{L} \varphi_{q_k}(k_i)\phi_{q_k}(k_j) \tag{1.54}$$

for some functions φ and ϕ. This means that the rank of this matrix is maximally L and addition of the rank one matrix δV can increase the rank to at most $L+1$. Therefore, for $\delta > 1$

$$\det(1+\hat{V}) = \det(1+\hat{V}+\delta\hat{V}) = 0. \tag{1.55}$$

The corresponding determinants with sine-kernels are not zero i.e. $\det(1+\hat{S}_\nu) \neq 0$. In this case we see that even though the difference between $\hat{V}$ and $\hat{S}_\nu$ is exponentially small, it cannot be neglected, contrary to the cases for $\delta \leq 1$. To address this issue we modify the definition of the tau function by considering summation over $\mathbf{k}$ in Eq. (1.6) instead of $\mathbf{q}$, namely

$$\tau_-(x) = \sum_{\mathbf{k}} |\langle \mathbf{k}|\mathbf{q}\rangle|^2 e^{-ix(\sum\limits_{i=1}^{N+1} k_i - \sum\limits_{i=1}^{N} q_i)}, \tag{1.56}$$

where the overlaps keep their form (1.10) but with the modified relation between $\nu(q)$ and $g(q)$, namely

$$e^{-g(q)} = e^{2\pi i\nu(q)} - 1. \tag{1.57}$$

In the thermodynamic limit this sum transforms into Fredholm determinants (see Appendix A)

$$\tau_-(x) = \det(1+\hat{V}_- + \delta\hat{V}_-) + (\Gamma - 1)\det(1+\hat{V}_-) \tag{1.58}$$

with

$$V_-(k,q) = \frac{e^{2\pi i\nu(k)} - 1}{4\pi} e^{ix(q+k)/2 + i(q-k)/2} \frac{E_-(k) - E_-(q)}{\sin\frac{k-q}{2}}, \tag{1.59}$$

$$\Gamma = \int\limits_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ixk}(1 - e^{-2\pi i\nu(k)}), \tag{1.60}$$

$$\delta V_-(k,q) = \frac{e^{2\pi i\nu(k)} - 1}{2\pi} \left(E_-(q) - i\Gamma/2\right)\left(E_-(k) + i\Gamma/2\right) e^{ix(q+k)/2}, \tag{1.61}$$

$$E_-(q) = \int\limits_{-\pi-i0}^{\pi-i0} \frac{dk}{4\pi} e^{-ixk}(e^{-2\pi i\nu(k)} - 1)\cot\frac{k-q}{2} + ie^{-ixq}. \tag{1.62}$$

For large $x > 0$, we notice that Γ is exponentially suppressed and $E_-(q) \approx ie^{-ixq}$, so $\tau_-(x)$ transforms into a generlized sine kernel Fredholm determinant Eq. (1.20) up to terms in exponentially small x. The corresponding asymptotic can be obtained in a way similar to $\delta < 0$, however instead of summation over "holes" q_a we will have summation over extra particles k_a. We demonstrate how it works for $\delta = 2$. In this case the set $\mathbf{q}$ consists of L elements and the set $\mathbf{k}$ of $L+1$ elements, which we parameterize by the omission one of the $L+2$ momenta from the all possible solutions of Eq. (1.7). Namely,

$$\mathbf{k}^{(a)} = \{k_1, \ldots, k_{a-1}, k_{a+1}, \ldots, k_{L+2}\}, \qquad a = 1, 2, \ldots, L+2. \tag{1.63}$$

The relative momentum of this state in the thermodynamic limit reads as

$$\Delta P = \sum_{k \in \mathbf{k}^{(a)}} k - \sum_{i=1}^{L} q_i = 2\pi - k_a - \int_{-\pi}^{\pi} \nu(q) dq. \tag{1.64}$$

The corresponding overlaps are given in Appendix C.2. Taking the thermodynamic limit we obtain the following presentation suited for the asymptotic analysis when $x \to +\infty$

$$\tau_-(x) = \mathcal{A}_- \int_{-\pi}^{\pi} \frac{dk}{2\pi} (e^{-2\pi i \nu(k)} - 1) \times \\ \exp\left(ik(x+2) + \int_{-\pi}^{\pi} \left(\nu(q) - \frac{q}{\pi}\right) \cot \frac{q-k-i0}{2} dq \right), \tag{1.65}$$

$$\mathcal{A}_- = \exp\left[-\frac{1}{2} \int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dk \left(\frac{\nu(q) - \nu(k) - (q-k)/\pi}{2 \sin \frac{q-k}{2}} \right)^2 \right]. \tag{1.66}$$

For $\delta > 2$ one can obtain similar determinant representation as for $\tau_-(x)$ as in Eq. (1.48).

Even though we have constructed $\tau_-(x)$ to address positive indices $\delta > 1$ it is possible to describe $\delta < 1$ by the previous choice of $g(k)$ Eq. (1.16) and considering $x < 0$.

1.4 Quantum XY spin chain and its correlation functions

In this section we consider an application of the general results obtained in the previous sections to the derivation of large distance asymptotics of thermal spin-

spin correlation functions of the quantum XY spin chain. The quantum XY spin chain in a transverse field is defined by the Hamiltonian [84, 90]

$$\mathbf{H}_{\mathrm{XY}} = -\frac{1}{2}\sum_{m=1}^{L}\left(\frac{1+\gamma}{2}\sigma_m^x\sigma_{m+1}^x + \frac{1-\gamma}{2}\sigma_m^y\sigma_{m+1}^y + h\sigma_m^z\right), \tag{1.67}$$

where a periodic boundary condition for the spin operators is assumed $\sigma_{L+1}^{\alpha} = \sigma_1^{\alpha}$, γ is an anisotropy parameter, and h is the strength of the magnetic field. The Hamiltonian $\mathbf{H}_{\mathrm{XY}}$ of XY model can be considered as an anisotropic deformation of the Hamiltonian $\mathbf{H}_{\mathrm{XX}}$ of XX model, corresponding to $\gamma = 0$. The Hamiltonian $\mathbf{H}_{\mathrm{XX}}$ can be diagonalized in two steps: Jordan–Wigner transformation to fermionic operators and a Fourier transform to momenta representation. To diagonalize the Hamiltonian $\mathbf{H}_{\mathrm{XY}}$ an additional Bogoliubov transformation is needed [84, 90, 75] specified by the angle $\theta(p)$:

$$e^{i\theta(p)} = \frac{h-\cos(p)+i\gamma\sin(p)}{\mathcal{E}(p)}, \qquad \mathcal{E}(p) = \sqrt{(h-\cos(p))^2+\gamma^2\sin^2(p)}. \tag{1.68}$$

Here $\mathcal{E}(p)$ stands for the spectrum of the effective Dirac fermions A_p, and the Hamiltonian $\mathbf{H}_{\mathrm{XY}}$ reduces to the free-fermionic one, namely

$$\mathbf{H}_{\mathrm{XY}} = \sum_p \mathcal{E}(p)\left(A_p^+ A_p - 1/2\right). \tag{1.69}$$

We skip the details of the fermions boundary conditions as they are not important in the thermodynamic limit. We focus on the following spin-spin correlation function at finite temperature

$$G(m) \equiv \langle\sigma_{m+1}^x\sigma_1^x\rangle_T = \frac{\mathrm{Tr}\left(\sigma_{m+1}^x\sigma_1^x e^{-\beta\mathbf{H}_{\mathrm{XY}}}\right)}{\mathrm{Tr}\left(e^{-\beta\mathbf{H}_{\mathrm{XY}}}\right)}. \tag{1.70}$$

It is the most interesting two point correlation function as the others are either trivial in the thermodynamic limit: $\langle\sigma_{m+1}^x\sigma_1^y\rangle_T = 0$, can be expressed in terms of elementary functions as $\langle\sigma_{m+1}^z\sigma_1^z\rangle_T$ (see Ref. [66]), or related to $G(m)$ after the change $\gamma\to-\gamma$ as $\langle\sigma_{m+1}^y\sigma_1^y\rangle_T$. We follow Ref. [75] to present $G(m)$ in the thermodynamic limit as Fredholm determinants ($m>0$):

$$G(m) = \det(1+\hat{W}+\widehat{\delta W}) - \det(1+\hat{W}), \tag{1.71}$$

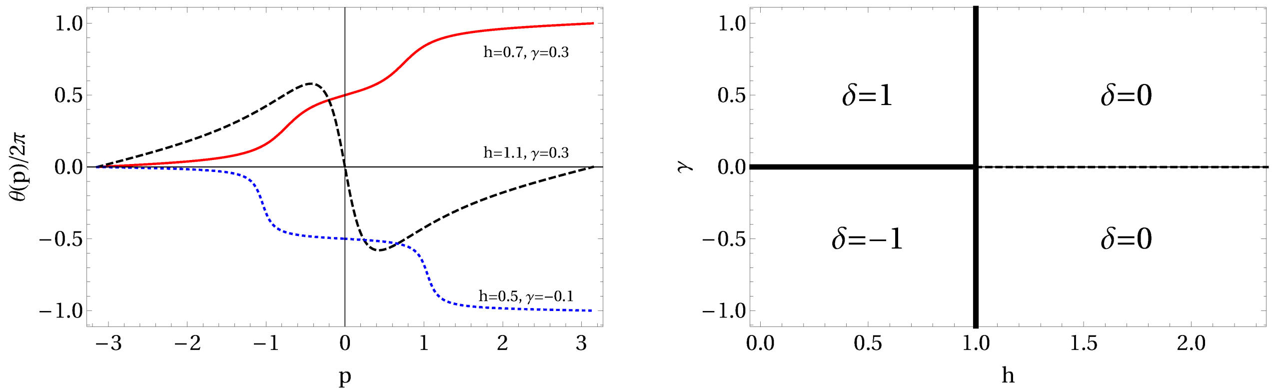


Figure 1.1: (left panel): the dependence of Bogoliubov angles on momentum for three different points in h–γ-plane: $h = 0.7$, $\gamma = 0.3$ ($\delta = 1$) – red solid, $h = 1.1$, $\gamma = 0.3$ ($\delta = 0$) – black dashed, $h = 0.5$, $\gamma = -0.1$ ($\delta = 0$) – blue dotted; (right panel): three regions in h–γ-plane corresponding to $\delta = \pm 1$ (ferromagnetic phase with $\gamma \gtrless 0$) and $\delta = 0$ (paramagnetic phase).

where the operators $\hat{W}$, $\widehat{\delta W}$ are integral operators on $L^2([-\pi, \pi])$ with the kernels given by

$$W(p,q) = -\frac{1}{\pi} e^{\frac{i(p-q)}{2}} \frac{\sin \frac{m(p-q)}{2}}{\sin \frac{p-q}{2}} \omega_F(q), \tag{1.72}$$

$$\delta W(p,q) = \frac{1}{\pi} \exp \frac{-im(p+q)}{2} \omega_F(q), \tag{1.73}$$

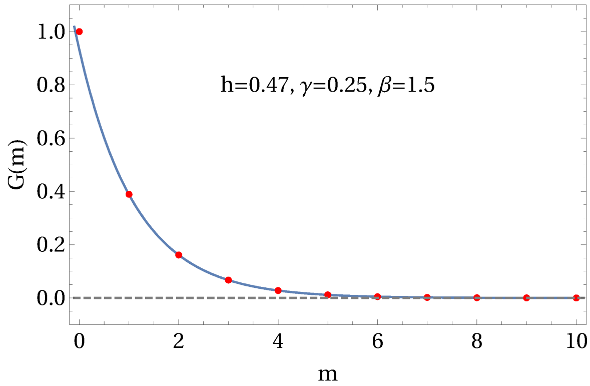
$$\omega_F(q) = \frac{1}{2}\left(1 - e^{i\theta(q)} \tanh \frac{\beta \mathcal{E}(q)}{2}\right). \tag{1.74}$$

In this form we immediately observe that $G(m)$ can be identified with $\tau_S(m)$ defined in Eq. (1.20) with the appropriate choice of $\nu(q)$, which can be read off from the prefactor in front of the sine-kernel

$$e^{2\pi i \nu(k)} = 1 - 2\omega_F(k) = e^{i\theta(k)} \tanh \frac{\beta \mathcal{E}(k)}{2}. \tag{1.75}$$

This way, to find the large m asymptotics we approximate $G(m)$ by $\tau(m)$ from Eq. (1.12) and use results for form factor series obtained in the previous sections. The analysis depends on the winding number $\delta = \nu(\pi) - \nu(-\pi)$, which can be read off from the following form of the phase shift

$$\nu(k) = \frac{\theta(k)}{2\pi} + \frac{1}{2\pi i} \log \tanh \frac{\beta \mathcal{E}(k)}{2}. \tag{1.76}$$



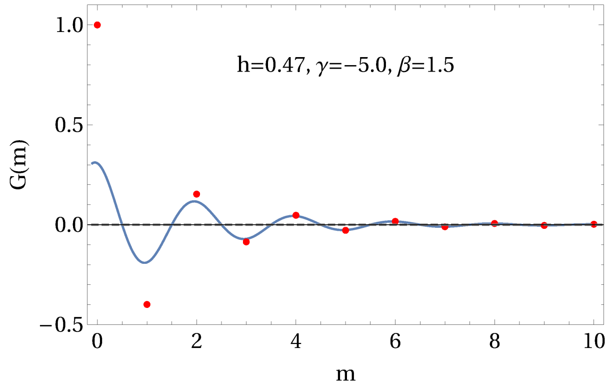


Figure 1.2: The exact values of the correlation function $G(m)$ (red dots) and its large distance asymptotics (blue solid curves). The left panel corresponds to $h = 0.47$, $\gamma = 0.25$, $\beta = 1.5$, and $\delta = +1$. The right panel corresponds to $h = 0.47$, $\gamma = -5.0$, $\beta = 1.5$, and $\delta = -1$.

The winding number is governed by the Bogoliubov angle $\theta(\pi) - \theta(-\pi) = 2\pi\delta$. The possible values of δ are $\delta = 0, \pm 1$ depending on the anisotropy parameter γ and the magnetic field h (see Fig. 1.1 for the typical behaviour of the Bogoliubov angle and the phase diagram).

Notice also that Eq. (1.76) implies that the integral entering the asymptotic formulas can be presented as

$$\int_{-\pi}^{\pi} dq\, \nu(q) = \pi\delta + \frac{1}{2\pi i}\int_{-\pi}^{\pi} dq\, \log\tanh\frac{\beta\mathcal{E}(q)}{2}. \tag{1.77}$$

In Fig. 1.2 we plot exact values for the correlation function $G(m)$ (red dots) and compare them with the asymptotic formulas written explicitly below (blue curves). We see that large m asymptotics gives reasonable approximation even for $m \sim 1$. In fact, to get any visual discrepancy we had to consider large negative anisotropies in the ferromagnetic phase ($\delta = -1$ and $\gamma < -1$). It turns out that in this case the asymptotic formulas for non-integer m acquire nonzero imaginary part, which is discarded in the plot. For integer points the imaginary part is equal to zero. Below we analyze each case separately and present analytical formulas for the asymptotics. These expressions turn out to be in accordance with the results of Ref. [66] but have a more compact form.

1.4.1 Paramagnetic phase $h > 1$ ($\delta = 0$)

We start our consideration with relatively large magnetic field $h > 1$. For zero temperature such values of h correspond to the paramangetic phase, while for finite temperature the corresponding $\nu(q)$ has zero winding number $\delta = 0$. This way, we use formula (1.35) to find asymptotic behavior of the correlation function $G(m)$ at large m, namely

$$G(m) = \tau_S(m) \approx -T_0(m) z_1^{-m-1} \mathfrak{S}(z_1)\, \mathrm{res}_{z=z_1}\, e^{-2\pi i \nu(k)}, \qquad z = e^{ik}. \tag{1.78}$$

where $T_0(m)$ and $\mathfrak{S}(z)$ are given by (1.30) and (1.33), respectively. The point z_1 is the position of the pole of $e^{-2\pi i\nu(k)}$ outside unit circle with minimal absolute value. To find z_1 we factorize

$$\begin{aligned} Q(z) = \mathcal{E}^2(k) = (h - \cos k)^2 + \gamma^2 \sin^2 k = \\ \frac{1-\gamma^2}{4z^2}(z - x_-)(z - x_+)(z - y_-)(z - y_+), \end{aligned} \tag{1.79}$$

$$x^{\pm} = \frac{h - \sqrt{h^2 + \gamma^2 - 1}}{1 \pm \gamma}, \quad y^{\pm} = \frac{h + \sqrt{h^2 + \gamma^2 - 1}}{1 \pm \gamma}, \quad (x^{\pm})^{-1} = y^{\mp}. \tag{1.80}$$

The exponent of the angle $\theta(k)$ of Bogoliubov transformation can also be presented in a factorized form, which leads to

$$e^{-2\pi i \nu(k)} = e^{-i\theta(k)} \coth \frac{\beta \mathcal{E}(k)}{2} = -\frac{2z}{1+\gamma} \frac{\sqrt{Q(z)} \coth \frac{\beta\sqrt{Q(z)}}{2}}{(z - x_+)(z - y_+)}. \tag{1.81}$$

It useful to present $\sqrt{Q(z)} \coth \frac{\beta}{2}\sqrt{Q(z)}$ as an infinite product

$$\sqrt{Q(z)} \coth \frac{\beta}{2} \sqrt{Q(z)} = \frac{2}{\beta} \frac{\prod_{n=1}^{\infty} \left(1 + \frac{\beta^2 Q(z)}{(2n-1)^2 \pi^2}\right)}{\prod_{n=1}^{\infty} \left(1 + \frac{\beta^2 Q(z)}{(2n)^2\pi^2}\right)}. \tag{1.82}$$

Notice that in such a form the branch cut singularities disappear manifestly. Moreover, the analysis of the poles of $e^{-2\pi i\nu(k)}$ is now a straightforward task, from which we conclude that the smallest (by the absolute value) pole outside the unit circle is $z_1 = y_+$ for all non-zero temperatures. Therefore using Eq. (1.81) and Eq. (1.82) we obtain

$$\mathrm{res}_{z=y_+}\, e^{-2\pi i\nu(k)} = -\frac{2}{\beta} \frac{y_+}{\sqrt{h^2 + \gamma^2 - 1}}. \tag{1.83}$$

Finally, taking into account (1.77) for $\delta = 0$, the asymptotics reads

$$G(m) \approx \mathcal{A} e^{-m/\xi}, \tag{1.84}$$

where

$$\xi^{-1} = \log y_+ - \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \, \log \tanh \frac{\beta \mathcal{E}(q)}{2}, \qquad y_+ = \frac{h + \sqrt{h^2 + \gamma^2 - 1}}{1 + \gamma}, \tag{1.85}$$

$$\mathcal{A} = \frac{2}{\beta \sqrt{h^2 + \gamma^2 - 1}} \times \\ \exp\left(-\frac{1}{2} \int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dp \left[\frac{\nu(q) - \nu(p)}{2 \sin \frac{q-p}{2}} \right]^2 + i \int_{-\pi}^{\pi} dq \, \nu(q) \frac{y_+ + e^{iq}}{y_+ - e^{iq}} \right). \tag{1.86}$$

The sign of the magnetic field h is irrelevant since it can be flipped by the conjugation of the Hamiltonian with σ^x acting in each site. Therefore below we consider $0 < h < 1$.

1.4.2 Ferromagnetic phase $h < 1$, $\gamma > 0$ ($\delta = 1$)

In the ferromagnetic phase $h < 1$ with positive anisotropy $\gamma > 0$ we use the asymptotics (1.24) and the integral (1.77) for $\delta = 1$ to obtain

$$G(m) \approx \mathcal{A} e^{-m/\xi}, \tag{1.87}$$

with

$$\xi^{-1} = -\frac{1}{2\pi} \int_{-\pi}^{\pi} dq \, \log \tanh \frac{\beta \mathcal{E}(q)}{2}, \tag{1.88}$$

$$\mathcal{A} = \exp\left(-\frac{1}{2} \int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dp \left[\frac{\nu(q) - \nu(p) - (q-p)/2\pi}{2 \sin \frac{q-p}{2}} \right]^2 \right). \tag{1.89}$$

For particular values of the parameters we plot exact correlation function $G(m)$ and its asymptotics (1.87) in the left panel of Fig. 1.2.

Note that even though formulas for the correlation length in different parameter regions Eq. (1.85) and Eq. (1.88) look different, the transition $h < 1$ and $h > 1$ is analytic in h. The same is true for prefactors $\mathcal{A}$ given by Eq. (1.86) and

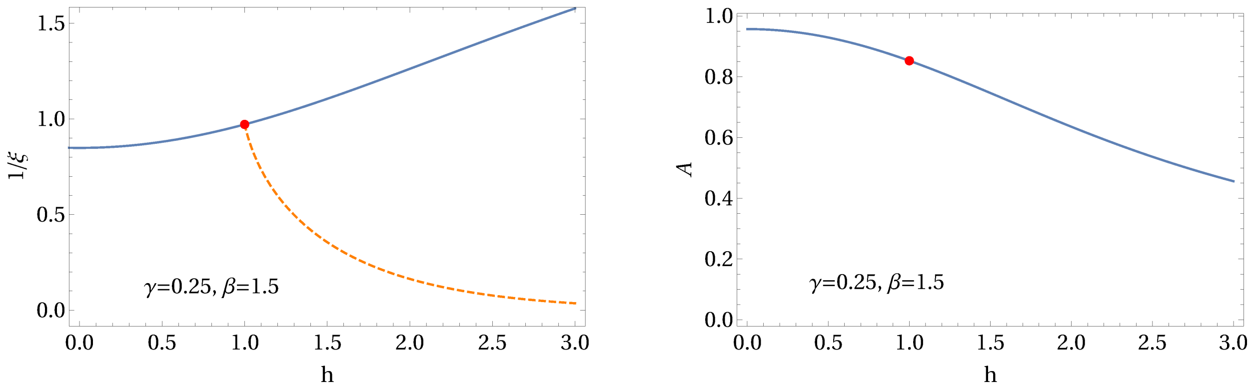


Figure 1.3: The inverse correlation length (left panel) and the prefactor (right panel) for different values of magnetic field h. Blue solid curves correspond to Eq. (1.88) [Eq. (1.89)] for $h < 1$ and Eq. (1.85) [Eq. (1.86)] for $h > 1$, for the left [right] panels, respectively. The orange line shows formal use of Eq. (1.88) for the region $h > 1$.

Eq. (1.89) (see the corresponding plots in Fig. 1.3). This reflects the fact that at finite temperature in one dimensional systems with short-range interactions phase transitions are absent and the physical observables are smooth functions of system parameters. This observation was used in Ref. [91] to obtain correct expressions for the correlation length and prefactor for the Ising model in the scaling limit.

1.4.3 Ferromagnetic phase $h < 1$, $\gamma < 0$ ($\delta = -1$)

In this region of parameters, the correlation function $G(m)$ is given by Eq. (1.48) for $n = 2$. We will need the large m asymptotics of $Y_{-1}(m)$, which for $h^2+\gamma^2 \neq 1$ is given by

$$Y_{-1}(m) \approx A_1 e^{-\varkappa_1 m} + A_2 e^{-\varkappa_2 m}, \tag{1.90}$$

where, as seen from Eq. (1.80),

$$\varkappa_1 = \log x_+, \qquad \varkappa_2 = \log y_+, \tag{1.91}$$

$$A_1 = \frac{2}{\beta}\frac{1}{\sqrt{h^2+\gamma^2-1}}\frac{1}{x_+}\mathfrak{S}_{-1}(x_+), \tag{1.92}$$

$$A_2 = -\frac{2}{\beta}\frac{1}{\sqrt{h^2+\gamma^2-1}}\frac{1}{y_+}\mathfrak{S}_{-1}(y_+), \tag{1.93}$$

$$\mathfrak{S}_{-1}(z) = \exp\left(i \int_{-\pi}^{\pi} dq \, \left(\nu(q) + \frac{\pi + q}{2\pi} \right) \frac{z + e^{iq}}{z - e^{iq}} \right). \tag{1.94}$$

Therefore Eq. (1.52) becomes

$$\begin{vmatrix} Y_{-1}(m) & Y_{-1}(m+1) \\ Y_{-1}(m-1) & Y_{-1}(m) \end{vmatrix} \approx \frac{16}{\beta^2(1-\gamma)^2} \mathfrak{S}_{-1}(x_+) \mathfrak{S}_{-1}(y_+) e^{-m(\log x_+ + \log y_+)}. \tag{1.95}$$

Finally, the large distance asymptotic for $G(m)$ following from (1.48) is

$$G(m) \approx \mathcal{A} e^{-m/\xi}, \tag{1.96}$$

where

$$\xi^{-1} = \log x_+ + \log y_+ - \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \, \log \tanh \frac{\beta \mathcal{E}(q)}{2}, \tag{1.97}$$

$$\mathcal{A} = \frac{16}{\beta^2(1-\gamma)^2} \mathfrak{S}_{-1}(x_+) \mathfrak{S}_{-1}(y_+) \times \exp\left(-\frac{1}{2} \int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dp \left[\frac{\nu(q) - \nu(p) + (q-p)/(2\pi)}{2 \sin \frac{q-p}{2}} \right]^2 \right). \tag{1.98}$$

In the case when $h^2 + \gamma^2 = 1$ we have $x_+ = y_+$ and the derivation is changed slightly (in particular, $Y_{-1}(m) \approx (B + Cm) e^{-m \log x_+}$) however the final formula for the asymptotic of $G(m)$ is the same. Notice that for non-integer values of m the right hand side of Eq. (1.96) becomes a complex valued function. We plot the typical behaviour of $G(m)$ and the real part of its asymptotics in (1.96) in the right panel of Fig. 1.2.

1.5 Relation to Toeplitz determinants

The traditional approach to the correlation functions in the XY spin chain is in presenting them via Toeplitz determinants [84, 66]. Asymptotic analysis of these structures can be performed by means of the Szegő theorem [92, 93] and its generalization[3] by Hartwig and Fisher [65]. Let us comment on how similar structures can appear within our effective form factors approach. In addition to tau functions (1.6) and (1.56) that contained different number of "particles" in bra- and ket- states, we define

$$\tau_0(x) = \sum_{\mathbf{q}} |\langle \mathbf{p}|\mathbf{q}\rangle|^2 e^{-ix\left(\sum\limits_{i=1}^{N} p_i - \sum\limits_{i=1}^{N} q_i\right)}, \tag{1.99}$$

where the quasi momenta q are solutions of $e^{iqL} = 1$, while p are solutions of the following equation

$$e^{ipL} = e^{-2\pi i \omega(p)}. \tag{1.100}$$

Here for convenience we have chosen a different notation for the phase shift. We focus on the case of non-positive winding numbers for this function i.e. $\omega(\pi) - \omega(-\pi) \le 0$. The corresponding form factors read

$$|\langle \mathbf{p}|\mathbf{q}\rangle|^2 = \frac{\prod\limits_{i=1}^{N} e^{g_\omega(p_i) - g_\omega(q_i)}}{\prod\limits_{i=1}^{N}(1 + \frac{2\pi}{L}\omega'(p_i))} \left(\prod_{i=1}^{N} \frac{\sin \pi\omega(p_i)}{L}\right)^2 \frac{\prod\limits_{i>j}^{N} \sin^2 \frac{p_i - p_j}{2} \prod\limits_{i>j}^{N} \sin^2 \frac{q_j - q_i}{2}}{\prod\limits_{i,j=1}^{N} \sin^2 \frac{p_i - q_j}{2}}. \tag{1.101}$$

The summation in Eq. (1.99) can be performed using techniques developed in Appendix (A), which together with the identification

$$e^{-g_\omega(p)} = e^{-2\pi i \omega(p)} - 1 \tag{1.102}$$

leads to the Fredholm determinant expression of τ_0

$$\tau_0(x) = \det(1 + \hat{V}_\omega), \qquad \hat{V}_\omega = \hat{S}_\omega + \hat{R}_\omega, \tag{1.103}$$

where

$$\hat{S}_\omega(p,q) = \frac{e^{2\pi i\omega(p)} - 1}{2\pi} \frac{\sin \frac{x(p-q)}{2}}{\sin \frac{p-q}{2}}, \tag{1.104}$$

$$\hat{R}_\omega(p,q) = \frac{e^{2\pi i\omega(p)} - 1}{4\pi} e^{-i(p+q)x/2} \frac{r_\omega(p) - r_\omega(q)}{\sin \frac{p-q}{2}}, \tag{1.105}$$

[3] Here we focus only on the smooth symbols with the only "singularity" given by the non-trivial winding number

$$r_\omega(k) = \int_{-\pi}^{\pi} \frac{dq}{4\pi}(e^{-2\pi i\omega(q)} - 1)e^{iqx} \cot \frac{q + i0 - k}{2}. \tag{1.106}$$

Notice that definitions of the kernels of $\hat{V}$ and $\hat{S}_\nu$ differ from their analogues introduced in Sec. 1.2 by the conjugation with diagonal matrices, which does not change the value of the determinant. Comparing overlaps (1.10) and (1.101) (see Appendix C.5) we conclude that imposing the following relation between $\nu(q)$ and $\omega(q)$

$$\omega(q) = \nu(q) - \frac{q+\pi}{2\pi} \equiv \nu_1(q), \tag{1.107}$$

we obtain exact equality for the tau functions, namely

$$\det\left(1 + \hat{V}_\nu + \delta\hat{V}_\nu\right) - \det\left(1 + \hat{V}_\nu\right) = \det\left(1 + \hat{V}_{\nu_1}\right). \tag{1.108}$$

Here the finite rank contribution is modified due the conjugation with the diagonal matrices

$$\delta V(p,q) = -\frac{e^{2\pi i\nu(p)} - 1}{2\pi} e^{-i(x+1)p/2} e^{-i(x-1)q/2}. \tag{1.109}$$

Similar relations can be obtained between $\tau_-(x)$ and $\tau_0(x)$ for $\delta > 1$. For large positive x, functions $r_\omega(x)$ are exponentially small, so Eq. (1.110) holds for the generalized sine-kernels $\hat{S}_\nu$.

$$\det\left(1 + \hat{S}_\nu + \delta\hat{V}_\nu\right) - \det\left(1 + \hat{S}_\nu\right) = \det\left(1 + \hat{S}_{\nu_1}\right). \tag{1.110}$$

In fact we can easily demonstrate that this relation is true for any positive integer x. To do so we will clarify the relation between Fredholm and Toeplitz determinants (*cf.* [94, 83]). It is convenient to slight deform of the kernel by the set of functions $a_0(p)$, $a_1(p)$, ..., $a_{x-1}(p)$

$$S^a_\nu(p,q) = \frac{e^{2\pi i\nu(p)} - 1}{2\pi} \sum_{n=0}^{x-1} a_n(p) e^{in(q-p)}. \tag{1.111}$$

For $a_i(q) = 1$ one can easily see that we recover the kernel $\hat{S}_\nu$ up to conjugation with diagonal matrices, which does not affect the value of the determinant

$$\det\left(1 + \hat{S}_\nu\right) = \det\left(1 + \widehat{S^a}\right)\Big|_{a_0 = a_1 = \ldots a_{x-1} = 1}. \tag{1.112}$$

Furthermore, we can treat $\widehat{S^a}$ as a product of two rectangular matrices

$$\widehat{S^a} = \mathcal{AB}, \qquad \mathcal{A}_{qn} = e^{iqn}, \qquad \mathcal{B}_{np} = \frac{e^{2\pi i\nu(p)} - 1}{2\pi} a_n(p) e^{-inp}. \tag{1.113}$$

Then using the fact that $\det(1+\mathcal{AB}) = \det(1+\mathcal{BA})$ we obtain a relation between the Fredholm determinant and determinant of matrix x by x, namely

$$\det\left(1+\widehat{S^a}\right) = \det_{0\le n,m\le x-1}\left(\delta_{nm}+T_{nm}\right), \tag{1.114}$$

$$T_{nm} = \int_{-\pi}^{\pi} \frac{dq}{2\pi} a_n(q)(e^{2\pi i\nu(q)}-1)e^{-i(n-m)q}. \tag{1.115}$$

For $a_n = 1$ the matrix T_{nm} transforms into the Toeplitz one, namely

$$\det\left(1+\widehat{S^a}\right) = \det_{0\le n,m\le x-1} c_{n-m}, \qquad c_k = \int_{-\pi}^{\pi} \frac{dq}{2\pi} e^{2\pi i\nu(q)}e^{-ikq}. \tag{1.116}$$

In order to account for the finite rank we notice that because rank-one contributions are at most linear in the determinant expansion, we can present

$$\det\left(1+\hat{S}_\nu+\delta\hat{V}_\nu\right) - \det\left(1+\hat{S}_\nu\right) = \frac{\partial}{\partial\alpha}\det\left(1+\hat{S}_\nu+\alpha\delta\hat{V}_\nu\right)\Big|_{\alpha=0}. \tag{1.117}$$

To account for the finite α one must choose $a_0(q) = 1-\alpha e^{-ixq}$ and $a_n(q) = 1$ for $n \geq 1$, therefore

$$\det\left(1+\hat{S}_\nu+\alpha\delta\hat{V}_\nu\right) = \det\left(1+\widehat{S^a}\right) =$$
$$\det\begin{pmatrix} c_0-\alpha c_x & c_{-1}-\alpha c_{x-1} & \dots & c_{-x+1}-\alpha c_1 \\ c_1 & c_0 & \dots & c_{-x+2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & \ddots & \cdot \\ \cdot & \cdot & & \cdot \\ c_{x-1} & c_{x-2} & \dots & c_0 \end{pmatrix}. \tag{1.118}$$

Since we are looking only to the terms linear in α, we can leave only terms that are proportional to α in the first row. Moreover we can replace this row with the last one. This way we obtain

$$\frac{\partial}{\partial\alpha}\det\left(1+\hat{S}_\nu+\alpha\delta\hat{V}_\nu\right)\Big|_{\alpha=0}=$$

$$(-1)^x\det\begin{pmatrix} c_1 & c_0 & \dots & c_{-x+2} \\ c_2 & c_1 & \dots & c_{-x+3} \\ . & . & & . \\ . & . & \ddots & . \\ . & . & & . \\ c_x & c_{x-1} & \dots & c_1 \end{pmatrix} = \det_{0\le n,m\le x-1}\tilde{c}_{n-m}, \quad (1.119)$$

where

$$\tilde{c}_k=-\int_{-\pi}^{\pi}\frac{dq}{2\pi}e^{2\pi i\nu(q)}e^{-i(k+1)q}=\int_{-\pi}^{\pi}\frac{dq}{2\pi}e^{2\pi i\nu_1(q)}e^{-ikq}. \quad (1.120)$$

Here we see the shift $\nu(q)\to\nu_1(q)$ as predicted from the finite size scaling of the form factors in Eq. (1.107). This shift together with Eq. (1.116) completes the proof of Eq. (1.110).

Let us also comment on how results of Sec. (1.3.3) reproduce Hartwig and Fisher asymptotic behaviour (Theorem 4 in Ref. [65]). As $\nu_\delta(q)$ has zero winding number we can expand it as

$$\nu_\delta(q)=\frac{-1}{2\pi i}\sum_{n=-\infty}^{\infty}k_n e^{iqn}. \quad (1.121)$$

Then the integral in the exponential in Eq. (1.50) can be evaluated as

$$-\int_{-\pi}^{\pi}dp\,\nu_\delta(p)\cot\frac{q-p+i0}{2}=\int_{-\pi}^{\pi}\frac{dp}{2\pi}\frac{e^{i(q+i0)}+e^{ip}}{e^{i(q+i0)}-e^{ip}}\sum_{n=-\infty}^{\infty}k_n e^{ipn}$$

$$=-k_0-2\sum_{n=1}^{\infty}e^{iqn}k_n. \quad (1.122)$$

In this derivation we used that $|e^{i(q+i0)}|<1$ and expanded the denominator as a geometric series. Substituting this result back into Eq. (1.50) we immediately see that $Y_\delta(x)=l_x$, where the Fourier modes l_m are defined through the relation

$$\exp\left(\sum_{n=1}^{\infty}\left(k_{-n}e^{-iqn}-k_n e^{iqn}\right)\right)=\sum_{m=-\infty}^{\infty}l_m e^{imq}. \quad (1.123)$$

Finally, expressing double integral in the asymptotic expression Eq. (1.48)

$$-\frac{1}{2}\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dp \left[\frac{\nu_\delta(q)-\nu_\delta(p)}{2\sin\frac{q-p}{2}}\right]^2 = \sum_{n=1}^{\infty} nk_n k_{-n}, \tag{1.124}$$

we obtain the statement of Theorem 4 in Ref. [65].

1.6 Conclusions

In this chapter we have introduced the form factors (overlaps) to simulate the static correlation functions for the states with finite entropy. The state was determined by the phase shift function $\nu(q)$. For the traditional approaches dealing with the finite entropy states is notoriously difficult but for our approach it is rather advantageous situation, since almost all available quantum numbers are occupied which tremendously simplifies the computation of form factor series. This allows us, in particular, to re-derive known asymptotics for the static two point correlators in the XY spin chain and present them in a more compact form. We hope that the simplicity of this approach will make it possible to obtain the full asymptotic expansion at large distances.

Apart from the thermal state we can apply our approach to the states resulting from the long time evolution after a quench [95, 96, 97, 45], to models of 1D anyons [98, 99, 100, 101, 102], or mobile impurity models [85, 103, 104]. This can be done by the appropriate modification of the phase shift function. We will discuss it elsewhere.

It is interesting to note that $\nu(q)$ is apparently connected with the auxiliary functions that appears in the Quantum Transfer Matrix (QTM) approach and specifies the Bethe roots for QTM [105, 106, 107, 108]. It would be interesting to completely clarify connection between these two approaches.

The correlation functions at zero temperature (entropy) can be formally accounted by the jump discontinuities in $\nu(q)$, which can also be treated by the form factor summation developed for the critical models [14, 16]. In this case the role of the lattice is not essential and the exponential asymptotic behaviour is expected to be replaced by a power-law, which can be obtained from the proper modification of the generalized sine-kernels (see section 9 in Ref. [109]). To address dynamical correlation functions we must modify appropriately the form factors and the spectral factor $e^{-i\sum k_i x} \to e^{-i\sum(k_i x - \epsilon(k_i)t)}$. The detailed constructions and extraction of the asymptotic behavior in this case is considered in the next chapter. However we can already anticipate that for the space-like region, i.e. when the saddle point of the expression $kx - \epsilon(k)t$ is outside the Brillouin zone, the asymptotic analysis remains largely unchanged, which can be immediately seen in the asymptotics of Ref. [82]. For the time-like region the main problem will be that a suitable $\nu(q)$ might have a jump discontinuity which leads to additional power-law behavior (*c.f.* Ref. [110]). Finally, there will be extra $1/\sqrt{t}$ terms connected to the saddle point contributions indicated by the non-linear Luttinger theory [17, 18, 19].

Chapter 2

Large-time and long-distance asymptotics of the thermal correlators of the impenetrable anyonic lattice gas

2.1 Introduction

In the previous chapter, we developed a method to deal with correlation functions in finite entropy states. This method allows one to derive the behavior of the correlation functions in free-fermionic models for the observables that can be expressed as Fredholm determinants of integrable kernels. In the previous chapter, we focused mostly on static correlation functions, and applied the method to the XY quantum chain.

In this chapter, we continue the development of the method of effective form-factors for dynamical correlation functions. As a model of interest, we choose one-dimensional impenetrable anyons on a lattice [67]. This model describes quantum particles with unusual statistics [111, 112, 113, 114, 67, 115], which can be realized experimentally in ultracold quantum gases confined in optical traps [116, 117, 118, 119, 120, 121, 122, 123, 124]. Furthermore, this type of models appears after the spin-charge separation in interacting systems of spinful fermions and spin chains (at certain values of the anyonic parameter) [125, 126, 127, 128, 129, 130, 131, 132, 133]. Similar determinants can also be obtained as the correlation functions of Wigner strings [134]. Also, they appear in the description of the mobile impurity propagating in the gas of free fermions [85, 135, 136, 137]. In the latter case the anyonic parameter can be identified with the total momentum of the system (at the infinite coupling).

The main idea of the effective form-factor approach is to replace computation of the correlation functions averaged over some ensemble to zero-temperature correlators with the appropriately modified phase shift. The correlation functions for one-dimensional impenetrable anyons can be presented as a linear combination of the Fredholm determinants [67]. Therefore, we may identify the phase shift comparing these determinants to the one that emerges from the summation of the effective form-factors. For the space-like region we can simplify the corresponding kernels for large-time and space separation and find the effective phase shift for all values of the quasi momenta. The time-like region is characterized by the critical points that separate different types of asymptotic behavior. So we can robustly find the effective phase shift only away from these points. Even though the vicinity of critical points where we do not know the solutions vanishes in the large-time limit, we cannot simply combine solutions in the different asymptotic regions into a single phase shift as the latter will be discontinuous. To tackle this problem, we have assumed the existence of the gluing regularization functions. While we have not been able to find them explicitly, we have demonstrated that they only affect the overall constant in the asymptotic expression of the Fredholm determinants.

The structure of this chapter is as follows. In Sec. 2.2.1 we define the anyonic model and recall its spectrum and the presentation of some correlation functions in terms of Fredholm determinants. In the subsection 2.2.2, for the reader's convenience, we collect the main results obtained in this chapter. In Sec. 2.3 we recall the effective form-factor approach and give two expressions for the τ function in the thermodynamic limit. In Sec. 2.4 the effective form-factor approach is applied to the derivation of the large-time and long-distance asymptotics of the dynamical correlation functions. We discuss separately space-like and time-like regimes. In Sec. 2.5 we summarize the main results of the chapter, compare with the known results in the literature, and discuss different possibilities for further research. Appendix D contains technical details of the asymptotic analysis of the form-factors with the regularized effective phase shift.

2.2 Model

2.2.1 Definition

The one-dimensional impenetrable lattice anyons on L sites can be described by the following Hamiltonian [67]

$$H = -\sum_{j=1}^{L} \frac{1}{2}(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) + h\sum_{j=1}^{L} a_j^\dagger a_j, \tag{2.1}$$

$$a_{L+1} = a_1, \qquad a_{L+1}^\dagger = a_1^\dagger. \tag{2.2}$$

The operator algebra is specified by the anyonic parameter $0 \leq \kappa \leq 1$ and reads as

$$a_j a_k^\dagger = \delta_{jk} - e^{-i\pi\kappa\epsilon(j-k)} a_k^\dagger a_j, \tag{2.3a}$$

$$a_j a_k = -e^{i\pi\kappa\epsilon(j-k)} a_k a_j, \tag{2.3b}$$

$$a_j^\dagger a_k^\dagger = -e^{i\pi\kappa\epsilon(j-k)} a_k^\dagger a_j^\dagger, \tag{2.3c}$$

here $\epsilon(j) = \text{sign}(j)$ and we prescribe that $\epsilon(0) = 0$.

The $\kappa = 0$ case corresponds to fermions, and $\kappa = 1$ describes operators in the Hilbert space of the impenetrable bosons. Note also that in the latter case, the Hamiltonian (2.1) can be identified with the Hamiltonian of the quantum XX spin chain after the mapping $a_j = \sigma_j^+$, $a_j^\dagger = \sigma_j^-$.

The spectrum of the Hamiltonian H can be found by means of the Bethe ansatz. The N-particle states are labeled by N momenta $\{p_1, p_2, \ldots, p_N\}$ from the set of L inequivalent solutions of the equation

$$e^{ipL} = e^{-i\pi\kappa(N-1)}. \tag{2.4}$$

The energies of such states are

$$E(\{p_1, p_2, \ldots, p_N\}) = \sum_{j=1}^{N} \varepsilon(p_j), \tag{2.5}$$

$$\varepsilon(p) = h - \cos p. \tag{2.6}$$

An interesting and non-trivial problem in the considered model is to analyze two-point correlation functions

$$G_-(x,t) = \frac{\mathrm{Tr}[e^{-\beta H} a^\dagger_{x+1}(t) a_1(0)]}{\mathrm{Tr}[e^{-\beta H}]}, \tag{2.7}$$

$$G_+(x,t) = \frac{\mathrm{Tr}[e^{-\beta H} a_{x+1}(t) a^\dagger_1(0)]}{\mathrm{Tr}[e^{-\beta H}]}. \tag{2.8}$$

It is easy to check the symmetry relations

$$G_\pm(-x,-t) = G_\pm(x,t)^*, \tag{2.9}$$

and also for $t = 0$

$$G_-(x,0) + e^{-i\pi\kappa\,\mathrm{sign}(x)} G_+(-x,0) = \delta_{x,0} \tag{2.10}$$

which allow us to consider only $t \geq 0$. In what follows we will restrict ourselves to the analysis of the correlator $G_-(x,t)$. An analogous analysis can be done for $G_+(x,t)$. It was shown that these correlators in the thermodynamic limit $L \to \infty$ can be written in terms of Fredholm determinants [67]. We will use the following equivalent representation for $G_-(x,t)$:

$$G_-(x,t) = \det(1 + \hat{W} + \delta\hat{W}) - \det(1 + \hat{W}), \tag{2.11}$$

where $\hat{W}$ and $\delta\hat{W}$ are integral operators on $[-\pi,\pi]$ with the kernels

$$W(p,q) = \frac{1}{2\pi} e_-(p) e_-(q) e^{\frac{i(p-q)}{2}} \frac{e(p) - e(q)}{\sin\frac{p-q}{2}}, \tag{2.12}$$

$$\delta W(p,q) = \frac{1}{2\pi} e_-(p) e_-(q), \tag{2.13}$$

$$n_F(p) = \frac{1}{e^{\beta\varepsilon(p)} + 1}, \tag{2.14}$$

$$e_-(p) = \sqrt{n_F(p)}\, e^{-ixp/2 + it\varepsilon(p)/2}, \tag{2.15}$$

$$e(p) = \sin^2\frac{\pi\kappa}{2} \fint_{-\pi}^{\pi} \frac{dq}{2\pi} e^{ixq - it\varepsilon(q)} \cot\frac{q-p}{2} + \frac{1}{2}\sin(\pi\kappa) e^{ixp - it\varepsilon(p)}. \tag{2.16}$$

Eq. (2.11) allows us to compute the correlation function $G_-(x,t)$ numerically. However, large-time and long-distance asymptotics of the correlation functions are hard to extract by numerical means due to the oscillatory behavior of integral kernels. In the present paper, we analyze these asymptotics analytically by means of the effective form-factor approach (see chapter 1).

2.2.2 Results

Before presenting an application of the effective form-factor method to the problem, for the reader's convenience we collect the main results obtained in this chapter: the asymptotic formulas for the correlation function $G_-(x,t)$ for large x and t with a fixed ratio $v = x/t$. To present the answer we will need the effective phase shift functions $\nu_\pm(q)$ defined as

$$\nu_\pm(q) = \pm\frac{1}{2\pi i}\log\left(1 + n_F(q)(e^{\pm i\pi\kappa} - 1)\right). \tag{2.17}$$

The asymptotic behavior of $G_-(x,t)$ depends essentially on v. The spacelike region is specified by the condition $v > 1$, and the asymptotics there reduces to the analysis of a single integral (2.60). Depending on the velocity, there are two additional regimes within the spacelike region: the, so-called, saddle-point-dominated regime $1 < v < v_c$, and the pole-dominated regime $v > v_c$. The critical velocity v_c separating these two regimes can be read off from Eq. (2.69).

The asymptotics for $1 < v < v_c$ reads

$$G_-(x,t) \approx C_1 K(x,t) t^{-1/2} e^{-x\log z_{\rm sp} + t\sqrt{v^2-1} + ith}, \tag{2.18}$$

where

$$z_{\rm sp} = iv + i\sqrt{v^2-1}, \tag{2.19}$$

$$K(x,t) = Z^2[\nu_+]e^{ix\int_{-\pi}^{\pi}\nu_+(q)dq}. \tag{2.20}$$

For $v \geq v_c$ a pole gives the leading contribution

$$G_-(x,t) \approx C_2 K(x,t) e^{-x\log z_0 + \frac{t\pi}{\beta}(1-\kappa)}, \tag{2.21}$$

where z_0 is given by

$$z_0 = h_0 + \sqrt{h_0^2 - 1}, \quad h_0 = h + \frac{i\pi}{\beta}(1-\kappa), \tag{2.22}$$

The prefactors $Z^2[\nu_+]$, C_1, and C_2 are constants on the rays of fixed v. Their explicit expressions are given by Eqs. (2.44), (2.71), and (2.73). Note, in the case of the saddle-point contribution there is an additional power factor $t^{-1/2}$ correcting the exponential decay of the correlation function. For $v < -1$ there are similar regions, and the asymptotics can be obtained from the above upon the replacement $\nu_+ \to \nu_-$.

For the timelike region, $0 < v < 1$, the asymptotics of the correlation function is given by

$$G_-(x,t) \approx R_\infty t^{-\delta_1^2-\delta_2^2} e^{i\int_{-\pi}^{\pi}(x-t\varepsilon'(q))\nu(q)dq} \times \left(\frac{a_1 e^{-ixq_1+it\varepsilon(q_1)}}{t^{\frac{1}{2}+\delta_1}} + \frac{a_2 e^{-ixq_2+it\varepsilon(q_2)}}{t^{\frac{1}{2}+\delta_2}} \right). \quad (2.23)$$

For a fixed v, constants a_1 and a_2 are given by Eq. (2.91), while the constant R_∞ still remains unknown. The critical momenta q_1 and q_2 are defined by

$$q_1 = \arcsin v, \qquad q_2 = \pi - \arcsin v, \quad (2.24)$$

the effective phase shift $\nu(q)$ is piecewise function

$$\nu(q) = \begin{cases} \nu_+(q) & \text{if } -\pi < q < q_1 \text{ or } q_2 < q \leq \pi, \\ \nu_-(q) & \text{if } q_1 < q < q_2, \end{cases} \quad (2.25)$$

and δ_1 and δ_2 are the magnitudes of jumps of $\nu(q)$ at critical momenta

$$\delta_1 = \nu_-(q_1) - \nu_+(q_1), \quad \delta_2 = \nu_+(q_2) - \nu_-(q_2). \quad (2.26)$$

Besides the expected exponential decay of the correlation function $G_-(x,t)$ we observe an additional power factor $t^{-\delta_1^2-\delta_2^2}$ depending on the parameters of the model.

2.3 Effective form-factor approach

2.3.1 Effective form-factors and tau function

In this section we recall the effective form-factor approach initiated in the chapter 1. To specify the effective form-factor we require two smooth periodic functions $\nu(k)$, $g(k)$. The first one is called the effective phase shift and defines the shifted set of momenta as solutions of

$$e^{ikL} = e^{-2\pi i\nu(k)}. \quad (2.27)$$

Here L is regarded as a system size. Since $\nu(k)$ is periodic, i.e., it has a zero winding number in terms of the chapter 1, the largest ordered set of the *shifted* momenta has L terms $\mathbf{k} = \{k_1, \dots, k_L\}$. Each k_i is a solution of (2.27). The *unshifted* momenta are solutions of

$$e^{iqL} = 1. \tag{2.28}$$

All momenta are considered up to the equivalence $k \sim k + 2\pi$, and it is convenient to choose them to have real parts in the Brillouin zone $[-\pi, \pi]$.

The effective form-factors are defined for the subsets of momenta $\mathbf{q}$ of the size $L-1$. Such subsets can be parameterized by the position of the "hole,"

$$\mathbf{q}^{(a)} = \{q_1, \dots, q_{a-1}, q_{a+1}, \dots, q_L\}, \quad a = 1, \dots, L. \tag{2.29}$$

The effective form-factor then reads

$$|\langle \mathbf{k}|\mathbf{q}^{(a)}\rangle|^2 = L^{1-2L} \prod_{j=1}^{L} \frac{e^{g(k_j)-g(q_j)} \sin^2 \pi\nu(k_j)}{1 + \frac{2\pi}{L}\nu'(k_j)} \times e^{g(q_a)} \mathrm{det}^2 D^a, \tag{2.30}$$

where $\mathrm{det} D^a$ is defined for $\mathbf{q}^{(a)}$ and is merely a trigonometric variation of the Cauchy determinant, in which the row corresponding to q_a is omitted and replaced with the line of 1

$$\mathrm{det} D^a = \begin{vmatrix} \cot \frac{k_1 - q_1}{2} & \dots & \cot \frac{k_L - q_1}{2} \\ \vdots & \ddots & \vdots \\ \cot \frac{k_1 - q_L}{2} & \dots & \cot \frac{k_L - q_L}{2} \\ 1 & \dots & 1 \end{vmatrix}, \tag{2.31}$$

As we deal only with the square of the determinant, we can set this line as the last one.

The tau (correlation) function is defined as a series over these form-factors

$$\tau(x,t) = \sum_{\mathbf{q}^a} |\langle \mathbf{k}|\mathbf{q}^a\rangle|^2 e^{-ix(P(\mathbf{k})-P(\mathbf{q}^a))+it(E(\mathbf{k})-E(\mathbf{q}^a))}. \tag{2.32}$$

Here we use notations for the momentum and energy of many-particle state $|\mathbf{q}\rangle$

$$P(\mathbf{q}) = \sum_{q \in \mathbf{q}} q, \quad E(\mathbf{q}) = \sum_{q \in \mathbf{q}} \varepsilon(q). \tag{2.33}$$

In chapter 1, we have demonstrated that in the thermodynamic limit $L \to \infty$ the tau function can be presented as a difference of two Fredholm determinants

$$\tau(x,t) = \det(1 + \hat{V} + \delta\hat{V}) - \det(1 + \hat{V}), \tag{2.34}$$

where $\hat{V}$ and $\delta\hat{V}$ are integral operators on $[-\pi, \pi]$ with kernels

$$V(p,q) = \frac{1}{2\pi} c_-(p) c_-(q) e^{\frac{i(p-q)}{2}} \frac{c(p) - c(q)}{\sin \frac{p-q}{2}}, \tag{2.35}$$

$$\delta V(p,q) = \frac{1}{2\pi} c_-(p) c_-(q), \tag{2.36}$$

$$c_-(p) = \sin \pi\nu(p) e^{-ixp/2 + it\varepsilon(p)/2 + g(p)/2}, \tag{2.37}$$

$$c(p) = \fint_{-\pi}^{\pi} \frac{dq}{2\pi} e^{ixq - it\varepsilon(q) - g(q)} \cot \frac{q-p}{2} + \cot \pi\nu(p) e^{ixp - it\varepsilon(p) - g(p)}. \tag{2.38}$$

This form allows us to relate the correlation function of anyons with the tau function for a special choice of $\nu(k)$ and $g(k)$. This relation will be described in the next section.

2.3.2 Finite size scaling

In this subsection we give an alternative formula for the tau function based on first taking the thermodynamic limit of the form-factors and then performing the summation. The obtained expressions will have a simple form convenient for asymptotic analysis.

We start by representing $\det D^a$ in a factorized form

$$\prod_{i=1}^{L} \frac{\sin^2 \pi\nu(k_i)}{L^2} \det{}^2 D^{(a)} = Z^2 \mathcal{Z}_a, \tag{2.39}$$

$$Z = \prod_{i=1}^{L} \prod_{j=1}^{i-1} \frac{\sin \frac{k_i - k_j}{2}}{\sin \frac{q_i - q_j}{2}}, \tag{2.40}$$

$$\mathcal{Z}_a = \sin^2 \frac{\pi\nu(k_a)}{L} \prod_{j \neq a}^{L} \frac{\sin^2 \frac{k_j - q_a}{2}}{\sin^2 \frac{q_j - q_a}{2}}. \tag{2.41}$$

Extracting the hole dependent factors the tau function (2.32) can be rewritten as

$$\tau(x,t) = L \cdot K(x,t) \cdot \sum_{a=1}^{L} e^{g(q_a)} \mathcal{Z}_a e^{-ixq_a + it\varepsilon(q_a)}, \tag{2.42}$$

where $K(x,t)$ is an a-independent part given by

$$K(x,t) = Z^2 e^{-ix(P(\mathbf{k}) - P(\mathbf{q})) + it(E(\mathbf{k}) - E(\mathbf{q}))} \times \prod_{j=1}^{L} \frac{e^{g(k_j) - g(q_j)}}{1 + \frac{2\pi}{L}\nu'(k_j)}. \tag{2.43}$$

The expressions Z^2 and $K(x,t)$ have a finite thermodynamic limit,

$$\log Z = -\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dk \left[\frac{\nu(q)-\nu(k)}{4\sin\frac{q-k}{2}}\right]^2, \tag{2.44}$$

$$\log K(x,t) = 2\log Z - \int_{-\pi}^{\pi} \nu(q) g'(q) dq + i\int_{-\pi}^{\pi} (x-\varepsilon'(q)t)\nu(q) dq. \tag{2.45}$$

The hole dependent factors are suppressed in the thermodynamic limit,

$$\mathcal{Z}_a \approx L^{-2} \sin^2 \pi\nu(q_a) \exp\left(-\fint_{-\pi}^{\pi} dq\, \nu(q) \cot\frac{q-q_a}{2}\right), \tag{2.46}$$

but the whole tau function (2.42) has a finite thermodynamic limit and can be presented as an integral,

$$\tau(x,t) = K(x,t)\int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{g(k)} \sin^2 \pi\nu(k) e^{-ixk+it\varepsilon(k)}$$
$$\times \exp\left(-\fint_{-\pi}^{\pi} dq\, \nu(q) \cot\frac{q-k}{2}\right). \tag{2.47}$$

Thus we have two alternative presentations of the tau function in the thermodynamic limit: Eq. (2.34) as a difference of Fredholm determinants, and Eq. (2.47) in terms of integrals. The first form is convenient for the identification with other models, and the second form is convenient for large x and t analysis.

2.4 Asymptotic behavior of anyonic correlation function

2.4.1 Anyons and effective fermions

To apply the method of effective form-factors for the large x and t asymptotics of the correlation function $G_-(x,t)$ given by (2.11), we have to find suitable functions $\nu(k)$ and $g(k)$. This can be done after the identification of the kernels in (2.11) and in (2.34). In this section, we focus on the case of $h>0$; the case

$h < 0$ can be considered similarly. Also, we restrict the value of the parameter of anyonic statistics to $0 \leq \kappa < 1$. The peculiarities with the limiting case $\kappa = 1$ corresponding to the quantum XX spin chain are briefly discussed in Sec. 2.5.

Equating $G_-(x,t) = \tau(x,t)$, we see that their integral kernels coincide if we choose $\nu(p)$ and $g(p)$ to satisfy the equations

$$c_-(p) = e_-(p), \quad c(p) = e(p). \tag{2.48}$$

The first equation gives a relation between $g(p)$ and $\nu(p)$

$$e^{-g(p)} = \frac{\sin^2 \pi\nu(p)}{n_F(p)}. \tag{2.49}$$

The second equation allows us to obtain an integral equation for $\nu(p)$

$$\int_{-\pi}^{\pi} \frac{dq}{2\pi} \left(\frac{\lambda_+(q)}{\tan \frac{q-p-i0}{2}} + \frac{\lambda_-(q)}{\tan \frac{q-p+i0}{2}} \right) e^{ixq - it\varepsilon(q)} = 0, \tag{2.50}$$

where we have denoted

$$\lambda_\pm(q) = \frac{e^{\pm 2\pi i \nu_\pm(q)} - e^{\pm 2\pi i \nu(q)}}{n_F(q)}, \tag{2.51}$$

$$e^{\pm 2\pi i \nu_\pm(q)} = 1 + n_F(q)(e^{\pm i\pi\kappa} - 1). \tag{2.52}$$

We can solve Eq. (2.50) asymptotically for large x and t. The solution has different forms for two different values of $v \equiv x/t$. We call $|v| > 1$ the spacelike region and $|v| < 1$ the timelike region. These names should not be confused with similar terms in the relativistic theory — there the spectrum is linear for all momenta. In our case the names come from the condition in which the function

$$\Phi(q) \equiv vq + \cos q \tag{2.53}$$

has (timelike) or does not have (spacelike) a critical point for $q \in [-\pi, \pi]$. This function is merely the phase $xq - \varepsilon(q)t$ up to rescaling by time and shift by the constant h.

2.4.2 Asymptotic behavior of the correlation function in the spacelike region

To treat Eq. (2.50), we first have to look at each of the integrals separately. It is useful to present them as

$$\int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{\lambda_\pm(q) e^{it\Phi(q)}}{\tan \frac{q-p\mp i0}{2}} = \lambda_\pm(p) \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{e^{it\Phi(q)}}{\tan \frac{q-p\mp i0}{2}} + \int_{-\pi}^{\pi} \frac{dq}{2\pi} \frac{(\lambda_\pm(q) - \lambda_\pm(p)) e^{it\Phi(q)}}{\tan \frac{q-p}{2}}. \tag{2.54}$$

If we assume that $\nu(q)$ does not become singular even in the asymptotic region, then in spacelike region the second term in RHS of Eq. (2.54) becomes exponentially small for large x and t. The remaining integral in (2.54) can be rewritten as

$$\int_{-\pi}^{\pi} \frac{dq}{2\pi i} \frac{e^{it\Phi(q)}}{\tan \frac{q-p\mp i0}{2}} = e^{it\Phi(p)}(F(p) \pm 1), \tag{2.55}$$

where

$$F(p) = e^{-it\Phi(p)} \fint_{-\pi}^{\pi} \frac{dq}{2\pi i} \frac{e^{it\Phi(q)}}{\tan \frac{q-p}{2}}. \tag{2.56}$$

For large $t > 0$ the function $F(p)$ can be approximated up to exponentially small terms as

$$F(p) \approx \operatorname{sign}(\Phi'(p)). \tag{2.57}$$

As the spacelike region is characterized by the absence of critical points of $\Phi(p)$ for $p \in [-\pi, \pi)$, we can put $\operatorname{sign}(\Phi'(p)) = \operatorname{sign} v$. This way, one of the two integrals in Eq. (2.50) is exponentially small due to (2.54) and (2.55), while the other allows us to find the effective phase shift for large $t > 0$

$$\nu(p) \approx \nu_{\operatorname{sign} v}(p), \tag{2.58}$$

where $\nu_\pm(p)$ are defined by Eqs. (2.52). We use this asymptotic solution and the relation (2.49) in (2.47) to obtain

$$\tau(x,t) = K(x,t) T(x,t) e^{ith}, \tag{2.59}$$

where $K(x,t)$ is given by Eq. (2.45) and $T(x,t)$ corresponds to the integral in (2.47), which after the change of variables $z = e^{ik}$, takes the following form

$$T(x,t) = \frac{1}{2\pi i} \oint_{C_>} \frac{dz}{z} \frac{e^{t\theta(z)} S(z)}{J(z) + e^{i\pi\kappa}}. \tag{2.60}$$

Here $C_>$ is a counterclockwise circle with a radius slightly larger than 1 and

$$\theta(z) = -v \log z - \frac{i}{2}(z + z^{-1}), \tag{2.61}$$

$$S(z) = \exp\left(i \int_{-\pi}^{\pi} dq\, \nu(q) \frac{z + e^{iq}}{z - e^{iq}} \right), \tag{2.62}$$

$$J(z) = \exp\left(\beta \left(h - \frac{z + z^{-1}}{2} \right) \right). \tag{2.63}$$

In what follows we consider $v > 1$; the other case, $v < -1$, can be considered in the same manner. To find large x and t asymptotics of $T(x,t)$, we deform the contour $C_>$ to the steepest-descent curve C_1 defined by

$$\operatorname{Im} \theta(z) = \operatorname{Im} \theta(z_{\rm sp}) = -\pi v/2 \tag{2.64}$$

going through the saddle point $z_{\rm sp}$

$$z_{\rm sp} = iv + i\sqrt{v^2 - 1}. \tag{2.65}$$

Deforming the contour we might cross the poles of the integrand, which can only appear from the denominator, since $S(z)$ is a holomorphic function for $|z| > 1$. This way, we get

$$T(x,t) = \frac{1}{2\pi i} \oint_{C_1} \frac{dz}{z} \frac{e^{t\theta(z)} S(z)}{J(z) + e^{i\pi\kappa}} - \sum_{n=-\infty}^{n_0} \operatorname{res}_{z=z_n} \frac{e^{t\theta(z)} S(z)}{z(J(z) + e^{i\pi\kappa})}, \tag{2.66}$$

where the points z_n are defined as

$$z_n = h_n + \sqrt{h_n^2 - 1}, \quad h_n = h + \frac{i\pi}{\beta}(2n + 1 - \kappa), \tag{2.67}$$

and n_0 is the maximal number of a pole, which was crossed in the deformation process. This number depends on the velocity v and can be found from the inequality

$$\arg z_{n_0} < \frac{\pi}{2} - \frac{h}{v} < \arg z_{n_0+1}. \tag{2.68}$$

Schematically, the contours $C_>$, C_1, and the positions of poles z_n are shown in Fig. 2.1.

The formula (2.66) allows one immediately to read off the asymptotic behavior. The residues produce exponentially decaying terms; the leading contribution is

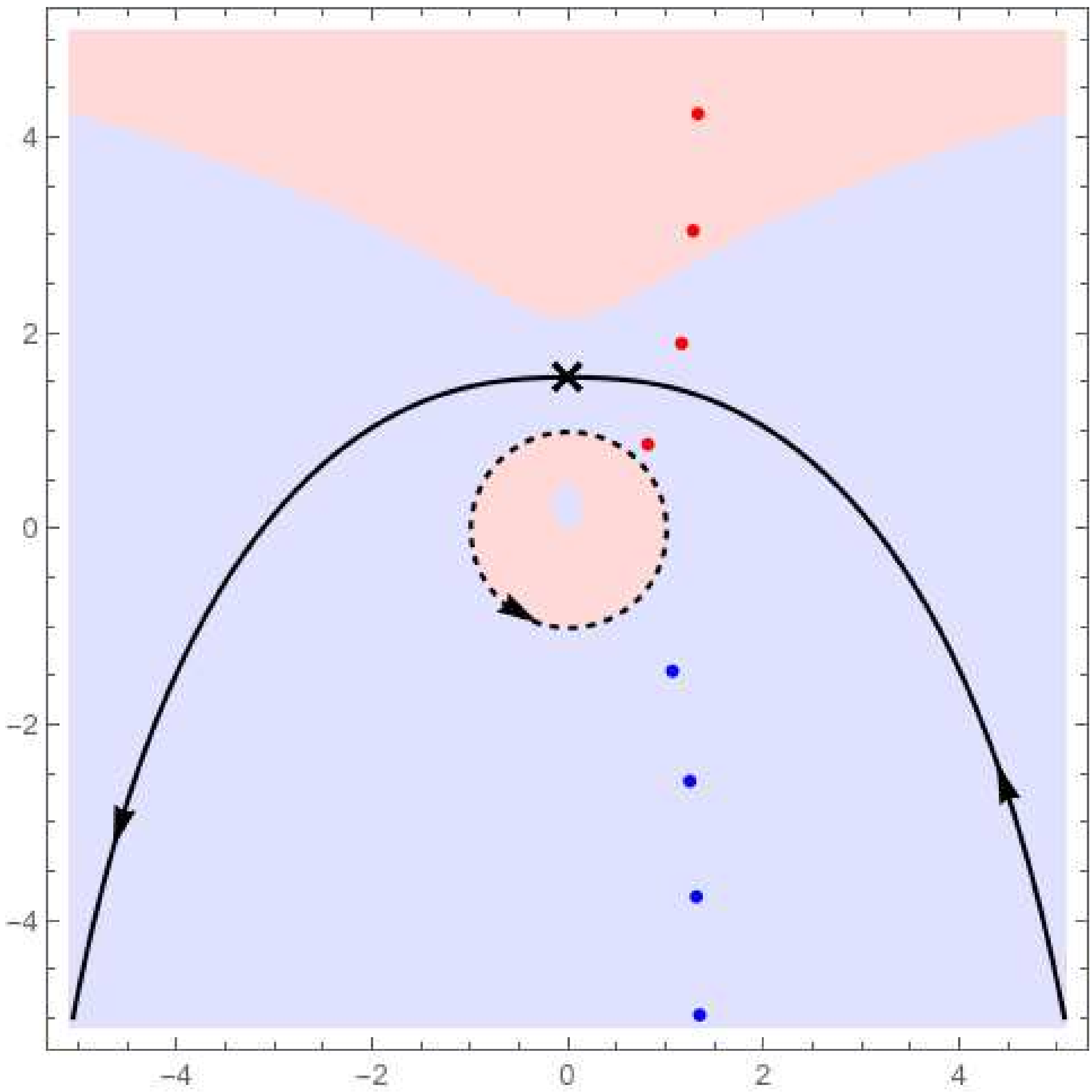


Figure 2.1: The integration contours (color online). The dashed circle corresponds to the initial contour of integration $C_>$. The black solid line represents the steepest descent contour C_1. The cross marks the position of the saddle point. Red and blue dots correspond to the poles z_n defined by Eq. (2.67) for non-negative and negative indices, respectively. The shaded areas show the regions of positive (pink) and negative (light blue) values of $\mathrm{Re}\,\theta(z)$, see Eq. (2.61).

given by the smallest real part $\mathrm{Re}\,\theta(z_n)$. For a wide range of the parameters of the model, we observed that this was achieved for the pole at z_0. Another type of contribution to the asymptotics comes from the saddle-point evaluation of the integral in (2.66). To find the overall leading contribution, we need to compare $\mathrm{Re}\,\theta(z_0)$ and $\mathrm{Re}\,\theta(z_{\mathrm{sp}})$. This leads to the equation for the critical velocity v_c separating two regimes,

$$v_c \log(v_c + \sqrt{v_c^2 - 1}) - \sqrt{v_c^2 - 1} = v_c \log|z_0| - \frac{\pi}{\beta}(1-\kappa). \tag{2.69}$$

For $v < v_c$, the saddle point is dominating and $T(x,t)$ is given by

$$T(x,t) \approx C_1 t^{-1/2} e^{-x\log z_{\mathrm{sp}} - \frac{it}{2}(z_{\mathrm{sp}} + z_{\mathrm{sp}}^{-1})}, \tag{2.70}$$

$$C_1 = \frac{S(z_{\mathrm{sp}})}{J(z_{\mathrm{sp}}) + e^{i\pi\kappa}} \frac{1}{\sqrt{2\pi\sqrt{v^2-1}}}. \tag{2.71}$$

For $v \geq v_c$, the pole gives the leading contribution

$$T(x,t) \approx C_2 e^{-x\log z_0 - \frac{it}{2}(z_0 + z_0^{-1})}, \tag{2.72}$$

$$C_2 = -\frac{2}{\beta}\frac{e^{-i\pi\kappa}}{z_0 - z_0^{-1}} S(z_0). \tag{2.73}$$

We also provide the simplified expression for $K(x,t)$,

$$\log K(x,t) \approx 2\log Z[\nu] + ix \int_{-\pi}^{\pi} \nu(q) dq, \tag{2.74}$$

where $\nu(q)$ is given by Eqs. (2.58) and (2.52), and $Z[\nu]$ is defined by Eq. (2.44).

Using the identification $G_-(x,t) = \tau(x,t)$, the asymptotic behavior of the correlation function $G_-(x,t)$ in the spacelike region can be found from Eq. (2.59)

$$G_-(x,t) \approx K(x,t) T(x,t) e^{ith}, \tag{2.75}$$

where $K(x,t)$ is given by Eq. (2.74), and $T(x,t)$ is given by one of‘ Eqs. (2.70), and (2.72) depending on the value of v. We compare these asymptotic expressions for the correlation functions with numerical evaluation of Fredholm determinants (2.11) in Fig. 2.2. We see that the asymptotics given by the integral (the red solid line), i.e. by the tau function, is hardly distinguishable from the true correlation function even for small x.

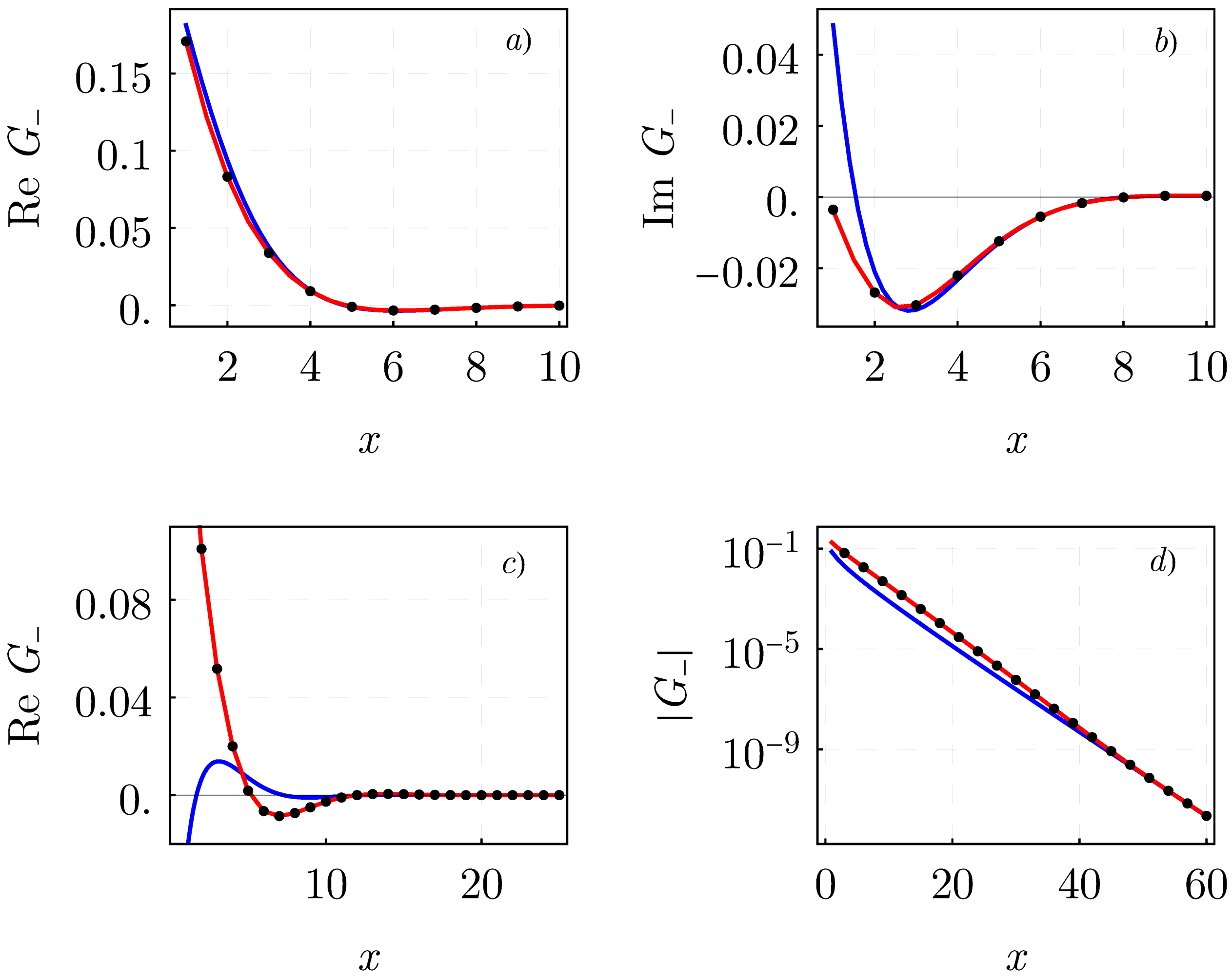


Figure 2.2: Asymptotic behavior of $G_-(x,t)$ for $\kappa = 0.6$, $h = 0.7$, $\beta = 2.3$. These parameters correspond to critical velocity $v_c \approx 1.676$. Black dots present $G_-(x,t)$ computed numerically from (2.11). Red lines present effective τ function (2.59) computed with (2.58). Blue lines present asymptotics of integrals in (2.59) given by Eqs. (2.70), (2.72). Panels a) and b) correspond to overcritical region $v = 2.5$. Panels c) and d) show real part and absolute value of $G_-(x,t)$ in the subcritical region $v = 1.3$, respectively.

2.4.3 Asymptotic behavior of correlation function in time-like region

Now let us try to apply the same reasoning for the timelike region, $|v| < 1$. In this case there are two critical points q_1 and q_2

$$\Phi'(q_i) = 0, \qquad q_i \in [-\pi, \pi), \tag{2.76}$$

therefore the approximation (2.57) naively gives rise to the solution

$$\nu(p) \approx \nu_{\operatorname{sign}\Phi'(p)}(p), \tag{2.77}$$

where $\nu_\pm(p)$ are defined by Eqs. (2.52). This is valid for all p lying far enough from the critical points. Indeed, the approximation (2.57) holds everywhere outside small vicinities of width $\sim t^{-1/2}$ around critical points q_1 and q_2.

It is very tempting to ignore these domains and approximate $\nu(p)$ as a truly discontinuous function, since we are interested in the large-t behavior. This procedure, however, is not consistent with the approximations made in Eq. (2.54) where we have discarded critical point contributions (the last integral). But even bigger problems appear when one tries to use discontinuous $\nu(p)$ for the asymptotic expression. For instance the double integral (2.44) is divergent for such a choice.

Therefore, we expect that the solution of Eq. (2.50) will have the following "regularized" form:

$$\nu(p) = A(p) + B(p)s(p), \tag{2.78}$$

where

$$A(p) = \frac{\nu_+(p) + \nu_-(p)}{2}, \qquad B(k) = \frac{\nu_+(p) - \nu_-(p)}{2}, \tag{2.79}$$

and the function $s(p)$ is a regularization of the sign function,

$$s(p) = f(\sqrt{t}\Phi'(p)), \tag{2.80}$$

with f being a smooth function satisfying

$$f(\pm\infty) = \pm 1. \tag{2.81}$$

So away from the critical points on a distance bigger than $O(1/\sqrt{t})$, we recover the solution (2.77). We demonstrate this schematically in Fig. 2.3. Notice that the regularization is needed only for the imaginary parts, and the real parts of $\nu_+(p)$ and $\nu_-(p)$ coincide. Now for the smooth $\nu(p)$ we can use all the results

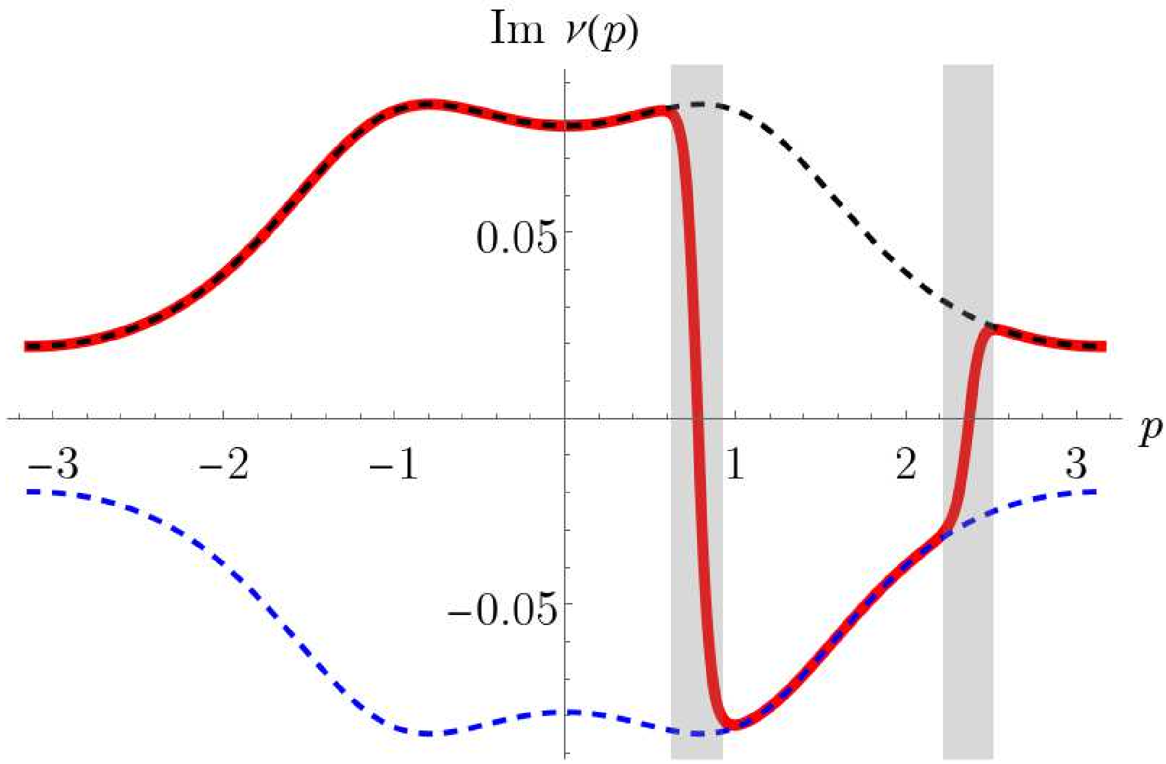


Figure 2.3: The schematic dependence of the effective phase shift $\nu(p)$. The black and blue dotted lines represent $\nu_+(q)$ and $\nu_-(q)$, respectively. The red lines shows the regularized expression for $\nu(p)$. The shaded rectangles show the regions where the transition between ν_+ and ν_- happens and the regularization is required to approximate $\nu(p)$. These regions are located near critical points q_1, q_2 and their widths are $O(t^{-1/2})$. We show only the imaginary part as the real part is continuous and does not require regularization.

from the previous sections. In particular, we can integrate Eq. (2.44) by parts to obtain

$$\log Z = \frac{1}{2}\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dk\, \nu'(q)\nu'(k) \log\left|\sin\frac{q-k}{2}\right|. \tag{2.82}$$

We can perform asymptotic analysis of this expression for large t and obtain

$$Z \approx t^{-\frac{1}{2}(\delta_1^2+\delta_2^2)}\, Z_{\text{reg}}, \tag{2.83}$$

where Z_{reg} is a t-independent factor depending on $s(p)$, and

$$\delta_1 = \nu_-(q_1) - \nu_+(q_1), \quad \delta_2 = \nu_+(q_2) - \nu_-(q_2). \tag{2.84}$$

Therefore the only regularization dependence remains in the overall constant prefactor. It is remarkable that the exponent of power law t-dependence of Z is universal [it does not depend on the regularization $s(p)$ for any f satisfying (2.81)]. These computations and the exact form for Z_{reg} are given in the Appendix.

Let us also discuss the asymptotic behavior of the remaining part of the tau function. In there we substitute already discontinuous $\nu(q)$. Namely, we analyze the integral

$$T(x,t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} n_F(k) e^{-it\Phi(k)} e^{-Y(k)}, \tag{2.85}$$

where

$$Y(k) = \fint_{-\pi}^{\pi} dq\, \nu(q) \cot\frac{q-k}{2}. \tag{2.86}$$

The function $Y(k)$ is logarithmically divergent at q_1 and q_2 because of the discontinuity of $\nu(q)$. It leads to power like singularities in the integrand of (2.85) which are integrable if $\operatorname{Re}\delta_j > -\frac{1}{2}$. In our case, $\nu_+(k)$ and $\nu_-(k)$ are conjugate to each other, rendering the real part of the effective phase shift continuous, $\operatorname{Re}\delta_j = 0$.

We separate a regular part $\tilde{Y}(k)$ of $Y(k)$ as

$$Y(k) = \tilde{Y}(k) + (\nu_-(k) - \nu_+(k)) \log\left(\frac{\sin\frac{q_1-k}{2}}{\sin\frac{q_2-k}{2}}\right)^2, \tag{2.87}$$

$$\begin{aligned}\tilde{Y}(k) = \int_{-\pi}^{q_1} dq\,(\nu_+(q) - \nu_+(k)) \cot\frac{q-k}{2} \\ + \int_{q_1}^{q_2} dq\,(\nu_-(q) - \nu_-(k)) \cot\frac{q-k}{2} \\ + \int_{q_2}^{\pi} dq\,(\nu_+(q) - \nu_+(k)) \cot\frac{q-k}{2}.\end{aligned} \tag{2.88}$$

Now all is prepared to find the asymptotic behavior of $T(x,t)$ for large x and t coming from the contributions of two critical points q_1 and q_2

$$T(x,t) \approx T_1 + T_2, \tag{2.89}$$

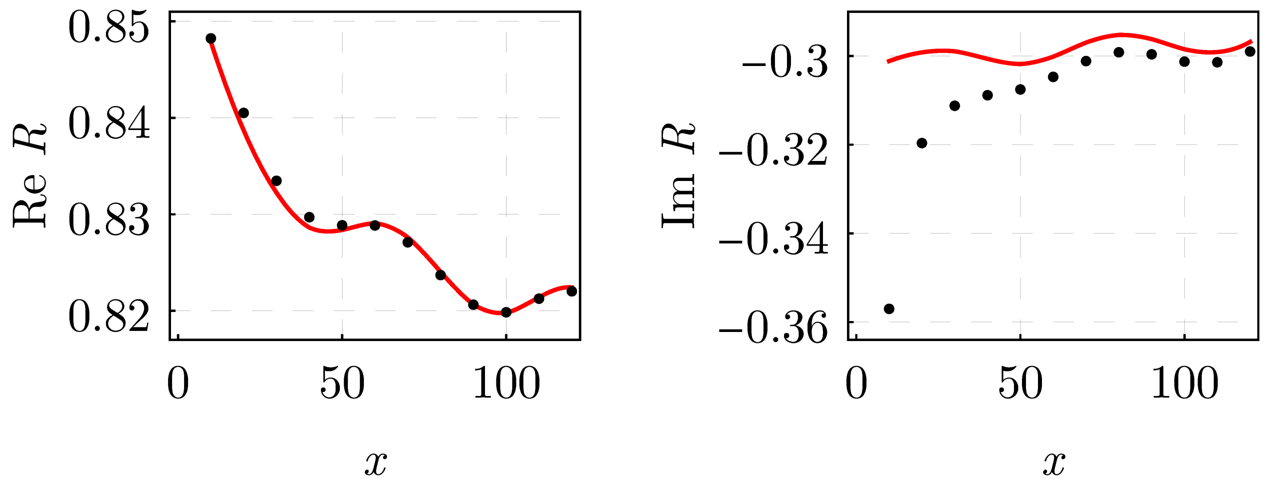


Figure 2.4: Real and imaginary part of $R(x,t)$ with $v = x/t = 0.5$, $\kappa = 0.6$, $h = 0.7$ and $\beta = 2.3$. Red line present $R(x,t)$ for which the integral $T(x,t)$ in Eq. (2.85) computed exactly. Black dots present $R(x,t)$ for which we use asymptotics of integral $T(x,t)$ given by Eq. (2.89).

where

$$T_j = a_j e^{-it\Phi(q_j)}, \tag{2.90}$$

$$a_j = \frac{n_F(q_j)}{2\pi} e^{-\tilde{Y}(q_j)} \left(2\sin\frac{q_2 - q_1}{2}\right)^{-2\delta_j} \left(\frac{it\Phi''(q_j)}{2}\right)^{-\frac{1}{2}-\delta_j} \Gamma\left(\frac{1}{2} + \delta_j\right). \tag{2.91}$$

The final formula for the asymptotics of the correlation function $G_-(x,t)$ is

$$G_-(x,t) \approx R_\infty T(x,t) t^{-\delta_1^2 - \delta_2^2} e^{iht} \exp\left(i\int_{-\pi}^{\pi} (x - t\sin q)\nu(q)dq\right), \tag{2.92}$$

where R_∞ is a constant different on each ray $v = x/t$ that additionally depends on the parameters κ, h, and inverse temperature β. To check this asymptotics, we plot in Fig. 2.4 the ratio $R(x,t)$ of $G_-(x,t)$ calculated numerically from (2.11) to the asymptotics from the right-hand side of Eq. (2.92) without R_∞. We observe that it approaches a constant value. The possible deviations are of order $O(1/\sqrt{t})$, which is consistent with our approximations made for the $\nu(k)$. It would be interesting to see if these corrections can be interpreted in terms of the non-linear Luttinger liquid paradigm [138, 18].

2.5 Conclusions

In this chapter, we found the asymptotics of dynamical correlation functions of anyonic gas with the parameter of anyonic statistics $0 \leq \kappa < 1$ using a effective form-factor approach. The main difficulty of this method is to find the phase-shift function $\nu(q)$ for effective fermions solving an integral equation. For large x and t we found approximate solutions for this integral equation that depend on the ratio $v = x/t$. For the spacelike region, $v > 1$, the solution $\nu(q)$ can be approximated by the smooth function $\nu_+(q)$. In this case, the asymptotics of the correlation function is given by asymptotic analysis of integrals producing the leading contribution either from a pole or from a saddle point. In the case of saddle-point contribution, there is an additional power factor correcting the exponential decay of the correlation function.

For the timelike region, $|v| < 1$, we approximate the solution $\nu(q)$ for a large finite t by a function having discontinuities at critical points and corresponding to the solution of the integral equation at $t = \infty$. Unfortunately, this approximate solution can not be used directly to find the asymptotics of the correlation function by the methods of the chapter 1, since the latter requires a smooth $\nu(q)$. For large finite t we consider a class of regularized $\nu(q)$ having the same limit at $t = \infty$ as the genuine solution. It is remarkable, that the regularized $\nu(q)$ lead to the same asymptotics up to a prefactor independent of t. This universal time dependence of asymptotics has an additional power-like factor to the exponential decay of the correlation function. The exponent of this power-like factor is related directly to the jumps of $\nu(q)$ at critical points. We hope that the use of a better approximation to $\nu(q)$ as a solution of the integral equation for a large finite t will fix the exact form of the constant prefactor. Further analysis of the correlation functions in the timelike region by the method of effective form-factors will be presented in future publications.

We believe that the appearance of the power-law corrections is universal and takes place in all dynamical correlation functions of quantum one-dimensional models at finite temperature (entropy) in timelike region. It was observed similar behavior for a continuum model [139]. Equivalent phenomena are present in XX spin chain [140, 106, 107]. Finally, quite unexpectedly, similar asymptotics appear also while describing large x and t behavior of the classical integrable systems [141, 142, 143]. Perhaps, it is related to the fact that the tau functions in such systems can be presented as Fredholm determinants, and the role of momentum distribution $n_F(q)$ is played by the reflection coefficient [144, 145]. We plan to investigate these models in the future.

The limiting case $\kappa = 1$ of the model corresponds to the quantum XX spin chain model studied intensively in the literature. Therefore, it is interesting to look at the limits of our results as $\kappa \to 1$ and compare with the known formulas. For the paramagnetic phase, $h > 1$, in timelike region the results for the asymptotics were obtained in [68] up to an overall constant depending on β and h. Our results have the same structure as a function of t. The ferromagnetic phase, $h < 1$, was studied in [140, 106, 107] in spacelike region and [140] in the timelike region. Unfortunately, the direct application of our approach is not possible due to the appearance of singularities of $\nu_\pm(q)$ at $q = \pm \arccos h$, where $\varepsilon(q) = 0$. We believe that these singularities can be properly resolved. But one needs to develop a more delicate limiting procedure, on which we hope to report in the nearest future.

An important ingredient in the derivation of asymptotics in [140, 68, 146] is the use of the fact that the correlation function satisfies differential-difference equations of Ablowitz–Ladik integrable system. It would be interesting to generalize this approach to the correlation functions with arbitrary anyonic parameter κ and determine the precise v dependence of R_∞ in Eq. (2.92).

Another important application of our approach is to use it to describe the scaling behavior of the correlation functions of the anyonic gas. One has to be able to reproduce results for the asymptotics obtained in [147, 148, 149]. Recently, using effective form-factors, the finite temperature tau function for the continuum case was investigated in [139].

Finally, an important generalization would be to the interacting case. Recently, the asymptotic behavior of the static one-body correlation function at zero temperature was derived for the interacting anyonic gas via the Luttinger liquid approach [150]. To reproduce this result, at least in the Tonks–Girardeau limit, we would have to take into account next to leading asymptotics. Indeed, the only way to reproduce zero temperature power-law behavior is first to recover finite T CFT predictions, which, roughly speaking, replace power-law as

$1/x^\Delta \to 1/(\sinh(Tx)/T)^\Delta$ [151]. In the expansion of this expression at large x one obtains not only the leading exponential but also a bunch of the subleading ones. One way to capture this could be in a more precise identification between integral kernels. Right now, we do not know how to generalize our methods to the fully interacting model, i.e. to the case when Fredholm presentation is not available. A perspective direction could be to derive it directly from the form-factor series [71].

Chapter 3

On Landauer–Büttiker formalism from a quantum quench

3.1 Introduction

In this chapter, we study the continuous bipartite system with an *arbitrary* defect localized around the middle of the system. We consider a bipartite quench protocol, in which initially the "right" part of the system is empty and the "left" part is filled up to some energy with fermions subjected to the local short-range potential $V_0(x)$, or distributed according to some probability (to model, for instance, the thermal initial state). After that, the dynamics of the whole is governed by the Hamiltonian with the local potential $V(x)$, which may, in principle, be different from $V_0(x)$. We compute the Full Counting Statistics (FCS) of the number of particles in the right part of the system. We derive an expression for FCS in the form of Fredholm determinant that is expressed via the Jost functions that correspond to the potentials V and V_0. This is an exact expression in the thermodynamic limit that describes both the transient dynamics and the formation of the non-equilibrium steady-state. We argue that in the absence of the bound states in the potential $V(x)$, the leading terms in the FCS are defined via the transmission coefficient of the potential $V(x)$ and are given by the Levitov–Lesovik formula [152, 153, 154] (with logarithmic corrections for zero temperature states). If two or more bound states are present in the system they affect even the properties of the steady state by introducing persistent oscillations with a frequency equal to the difference of energies between the bound states. Moreover, the amplitude of these oscillations depends on the Jost functions of the potential $V_0(x)$, this way retaining the memory of the initial state. This phenomenon can be observed already on the level of the current, where even for the constant bias the persistent oscillations are present on top of the constant Landauer–Büttiker contribution. Similar dependencies of the initial correlation in the case when bound states are present in the system were observed in [155, 156]. This effect seems to be overlooked in the traditional approach (see for instance footnote 54 in [157]).

The chapter is organized as follows. In Section 3.2 we recall definitions of the scattering data, the Jost states and adopt notations for one-dimensional systems. In Section 3.3 we formulate the problem and present the main results. The outline of the derivation of the main results is presented in Sections 3.4 and 3.5. In Section 3.4 we describe a construction of the wave functions in the finite system and in Section 3.5 we discuss how to obtain the kernel for the Fredholm determinant. Section 3.6 contains derivation of the Landauer–Büttiker expression for the current and its modification in the case when multiple bound states are present in the systems. A short summary and outlook are presented in Section 3.7. Appendices E, F, G and H deal with some details of the derivations and contain scattering data for a few exemplary potentials.

3.2 General properties of scattering

In this section we briefly remind some general notions of the one-dimensional scattering on the local potential $V(x)$. The eigenvalue problem satisfies the Schrodinger equation

$$H_V \Psi = \left(-\frac{d^2}{dx^2} + V(x) \right) \Psi = E\Psi. \tag{3.1}$$

The locality means that the potential vanishes fast enough as $|x| \to \infty$. For all practical purposes we assume that the potential is nonzero only in the finite domain $|x| < \xi$. This way, for $|x| > \xi$ the wave functions that correspond to the energy $E = k^2$ are the plane waves $e^{\pm ikx}$. So for every real $k \neq 0$ there exists a two-dimensional space of solutions. The typical basis in this space can be conveniently described by the Jost states ψ_k, φ_k defined by their asymptotic behavior, namely

$$\psi_k(x) = e^{-ikx} + o(1), \qquad x \to +\infty, \tag{3.2}$$

$$\varphi_k(x) = e^{-ikx} + o(1), \qquad x \to -\infty. \tag{3.3}$$

For a real potential these states are connected to their complex conjugated counterparts as $\psi_{-k}(x) = \bar{\psi}_k(x)$, $\varphi_{-k}(x) = \bar{\varphi}_k(x)$. If additionally the potential is symmetric $V(x) = V(-x)$, then $\psi_k(-x)$ and $\varphi_k(-x)$ are still eigenfunctions. Considering the asymptotic behavior one can conclude that in this case $\psi_k(-x) = \bar{\varphi}_k(x)$. Using (3.2) we see that the Jost solutions satisfy the following integral equations

$$\psi_k(x) = e^{-ikx} - \int_x^\infty \frac{\sin(k(x-y))}{k} V(y)\psi_k(y)dy, \tag{3.4}$$

$$\varphi_k(x) = e^{-ikx} + \int_{-\infty}^x \frac{\sin(k(x-y))}{k} V(y)\varphi_k(y)dy. \tag{3.5}$$

As both Jost solutions form a basis they are connected by the linear transformation, the transfer matrix,

$$\begin{pmatrix} \varphi_k(x) \\ \bar{\varphi}_k(x) \end{pmatrix} = \mathcal{T}(k) \begin{pmatrix} \psi_k(x) \\ \bar{\psi}_k(x) \end{pmatrix}, \qquad \mathcal{T}(k) = \begin{pmatrix} a_k & b_k \\ \bar{b}_k & \bar{a}_k \end{pmatrix}. \tag{3.6}$$

Note that for a real potential $a_{-k} = \bar{a}_k$, $b_{-k} = \bar{b}_k$, while for a symmetric potential b_k is purely imaginary.

Considering the Wronskian of the eigenvalue problem (3.2) we conclude that the transfer matrix is unimodular

$$\det \mathcal{T}(k) = |a_k|^2 - |b_k|^2 = 1. \tag{3.7}$$

The transfer matrix $\mathcal{T}$ can be repacked into the S-matrix [158] as follows

$$S = \frac{1}{a_k} \begin{pmatrix} -\bar{b}_k & 1 \\ 1 & b_k \end{pmatrix}. \tag{3.8}$$

The unimodularity condition (3.7) means the unitarity for S-matrix $SS^+ = 1$. The transmission and the reflection coefficients are defined as the squared absolute values of the off-diagonal and diagonal components of the S-matrix, respectively,

$$T(E) = \frac{1}{|a_k|^2}, \qquad R(E) = \frac{|b_k|^2}{|a_k|^2}. \tag{3.9}$$

Here we present them as the functions of energy $E = k^2$. The unitarity (3.7) guarantees that $T(E) + R(E) = 1$.

The coefficient a_k can be analytically continued to the upper half plane where it might have zeroes that correspond to the bound states. They are purely imaginary $k = i\varkappa$ so the corresponding energy is negative $E = -\varkappa^2$. In fact the analytic properties allow one to present (see for instance [159])

$$a_k = \prod_{n=1}^{N} \frac{k - i\varkappa_n}{k + i\varkappa_n} \exp\left(\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(1 + |b_q|^2)}{q - k - i0} dq \right). \tag{3.10}$$

To describe the wave function of a bound state we can use either $\varphi_k(x)$ and $\bar{\psi}_k(x)$ as both these functions can be analytically continued to the upper half plane. In fact, it turns out that they are proportional $\varphi_{i\varkappa}(x) = b_\varkappa \bar{\psi}_{i\varkappa}$. Taking into account the definition of transfer matrix (3.6) this relation is hardly surprising and $b_\varkappa$ can be considered as an analytic continuation of the b_k, however, contrary to a_k such continuation is not always possible, and the coefficient $b_\varkappa$ should be considered as additional scattering data.

Finally, let us comment on the normalization conditions of the continuous spectrum. Similar to [159] we conclude that

$$\int_{-\infty}^{\infty} dx \varphi_k(x) \bar{\psi}_q(x) = a_q \delta(k - q). \tag{3.11}$$

Theqrefore the Green's function $G(x, y, t)$ defined as a solution of the Schrodinger equation in x variable with the initial condition $G(x, y, t = 0) = \delta(x - y)$, can be presented as

$$G(x, y, t) = \int_C \frac{dk}{2\pi} \frac{\varphi_k(x)\bar{\psi}_k(y)}{a_k} e^{-itE_k}. \tag{3.12}$$

As for the continuum spectrum the contour C goes along the real line. We notice however that the integrand can be analytically continued in the upper half plane. Moreover, in this form we can easily take into account also contributions from the bound states. To do so the contour C should run above all positions of zeroes of a_k in the upper half plane (see figure 3.2 below). Below we re-derive this presentation using wave functions in the box (hard-wall boundary conditions), and demonstrate how to express full counting statistics via the scattering data and Jost solutions.

3.3 Quench protocol

The scattering states introduced in the previous section describe an infinite system. To correctly formulate transport problem we consider open (hard-wall) boundary conditions placed at $x = \pm R$, perform computations at finite R, and send $R \to \infty$ in the end of the computation. At the initial moment of time only the left part of the system $x < 0$ is filled. Meaning that the single particle wave functions Λ_q are non-zero only in the interval $x \in [-R, 0]$, more formally

$$-\frac{d^2\Lambda_q}{dx^2} + V_0(x)\Lambda_q = q^2\Lambda_q, \qquad \Lambda_q(0) = \Lambda_q(-R) = 0. \tag{3.13}$$

The post-quench wave functions satisfies

$$-\frac{d^2\chi_k}{dx^2} + V(x)\chi_k = k^2\chi_k, \qquad \chi_k(-R) = \chi_k(R) = 0. \tag{3.14}$$

The initial N-particle state of the system $|\text{in}\rangle$ is given in a Fock space by an ordered set of momenta $q_1 < q_2 < \cdots < q_N$. Formally, it can be presented as a wedge product

$$|\text{in}\rangle = \Lambda_{q_1} \bigwedge \Lambda_{q_2} \cdots \bigwedge \Lambda_{q_N}, \tag{3.15}$$

which in the coordinate space corresponds to a single Slater determinant. The case of the statistical ensemble in the $N \to \infty$ limit can be described by taking the typical distribution of q_i. To characterize many body dynamics we consider full counting statistics (FCS). It can be written as

$$\mathcal{F}(\lambda, t) = \langle \text{in}|e^{itH}e^{\lambda N_R}e^{-itH}|\text{in}\rangle = \langle \text{in}|e^{\lambda\int_0^t d\tau J(\tau)}|\text{in}\rangle, \tag{3.16}$$

where N_R is number of particles in right part of the system and $J(\tau)$ is the current through the point $x = 0$. Introducing the resolution of the unity, we can formally present FCS as a form factor series

$$\mathcal{F}(\lambda, t) = \sum_{\mathbf{k},\mathbf{p}} \langle \text{in}|\mathbf{k}\rangle\langle \mathbf{k}|e^{\lambda N_R}|\mathbf{p}\rangle\langle \mathbf{p}|\text{in}\rangle e^{it(E_{\mathbf{k}}-E_{\mathbf{p}})}. \tag{3.17}$$

Here $|\mathbf{k}\rangle$ and $|\mathbf{p}\rangle$ are many-body states of the form (3.15). Theqrefore the overlaps and the matrix elements are the determinants of the Cauchy type matrices.

Due to the free fermionic structure of the initial state (3.15) the FCS can be presented as

$$\mathcal{F}(\lambda, t) = \det X_{ab}, \tag{3.18}$$

with indices a and b corresponding to the momenta in the initial state $|\text{in}\rangle$, and the matrix elements are

$$X_{ab} = \delta_{ab} + (e^{\lambda} - 1) \sum_{k,p} \frac{(\Lambda_a, \chi_k)(\chi_k, P_> \chi_p)(\chi_p, \Lambda_b)}{\sqrt{(\Lambda_a, \Lambda_a)}(\chi_k, \chi_k)(\chi_p, \chi_p)\sqrt{(\Lambda_b, \Lambda_b)}} e^{it(E_k - E_p)}. \tag{3.19}$$

Here $P_>$ is a projector on the right part of the system i.e. $x \in [0, R)$. This formula can be obtained from (3.17) using some variant of the Cauchy–Binet formula (the product of determinants is the determinant of product of matrices). Our goal is to present (3.18) in the thermodynamic limit as a Fredholm determinant of some trace-class operator. Namely, we present

$$X_{ab} = \delta_{ab} + \frac{\pi}{R} K(q_a, q_b) + o(1/R) \tag{3.20}$$

so that FCS in the thermodynamic limit $R \to \infty$ transforms into a Fredholm determinant

$$\mathcal{F}(\lambda, t) = \det X \to \det\left(1 + \rho \hat{K}\right), \tag{3.21}$$

where ρ in the density of the initial state and the operator $\hat{K}$ acts on the integrable functions $L^2(\mathbb{R})$ via the convolution with the kernel $K(q, q')$, namely

$$\hat{K} f(q) = \int K(q, q') f(q') dq'. \tag{3.22}$$

We compute this kernel in Section 3.5. It can be presented as

$$K(q, q') = K_0(q, q') + \delta K(q, q'), \tag{3.23}$$

where

$$K_0(q, q') = \frac{e^{\lambda} - 1}{\pi} \sigma(q, q') \frac{\sin \frac{t(E_q - E_{q'})}{2}}{E_q - E_{q'}} \tag{3.24}$$

with

$$\sigma(q, q') = \frac{i|\Phi_q(0)||\Phi_{q'}(0)|}{\Phi_q(0)\Phi_{q'}(0)\bar{a}_q a_{q'}} \left(\bar{\psi}_{q'}(0)\partial_x \psi_q(0) - \psi_q(0)\partial_x \bar{\psi}_{q'}(0).\right) \tag{3.25}$$

Here ψ_k are Jost solutions defined by equation (3.4) and by $\Phi_k(x)$ we denote the Jost solution equation (3.5) on the potential V_0. The expression for δK can be found in Section 3.5. It contains, in particular, contributions from the bound states if they are present in the spectrum of $V(x)$. We see that the kernels are expressed via the scattering data and the Jost solutions. The separation on K_0 and δK is done to facilitate the large t asymptotic analysis. Namely, in this limit δK contains only oscillating terms, while formally K_0 tends to a delta function. For this reasoning we can heuristically argue that the leading contribution to the FCS will be given by K_0 and δK will results in smooth prefactor for FCS. For a specific lattice system this effect was observed in [160]. Moreover, since $\sigma(q,q')$ is a smooth function we can replace it with diagonal values $\sigma(q,q') \to \sigma(q,q)$. Further, taking into account that the Wronskian $\bar{\psi}_q(x)\partial_x\psi_q(x) - \psi_q(x)\partial_x\bar{\psi}_q(x)$ does not depend on x, which can be checked by the immediate differentiation. We evaluate it at $x \to -\infty$ and arrive to the conclusion that $\sigma(q,q) = 2q/|a_q|^2$. This allows us to transform the kernel to act on the energy space instead of momentum. This way, we obtain a Fredholm determinant of the generalized sine-kernel type

$$\mathcal{F}(\lambda,t) \approx \tilde{C}(\lambda,t)\det\left(1 + \frac{e^\lambda - 1}{\pi}\rho(E)T(E)\frac{\sin\frac{t(E-E')}{2}}{E-E'}\right). \tag{3.26}$$

Here we have written a kernel of the integral operator. The prefactor $\tilde{C}(\lambda,t)$ appeared due to discarding δK. Notice that in this form all information about the Jost function disappears and only the transmission coefficient $T(E)$ for the post-quench potential remains. Large t asymptotic behavior of the Fredholm determinant can be easily found either by solving the corresponding Riemann–Hilbert problem [109, 81, 161] or using the effective form factors developed in the chapters 1, 2. It is worth to mention the paper [139], where the effective form factor approach was applied for the case of continuum systems. For the smooth distribution $\rho(E)$ the result reads

$$\mathcal{F}(\lambda,t) \approx C(\lambda,t)\mathcal{F}_s(\lambda,t) \tag{3.27}$$

with

$$\log\mathcal{F}_s(\lambda,t) = \frac{t}{2\pi}\int \log(1 + (e^\lambda - 1)\rho(E)T(E))dE \equiv it\int \nu_\lambda(E)dE. \tag{3.28}$$

The prefactor $C(\lambda,t)$ contains both $\tilde{C}(\lambda,t)$ and the constant prefactors from the asymptotic expression for the Fredholm determinant. When bound states

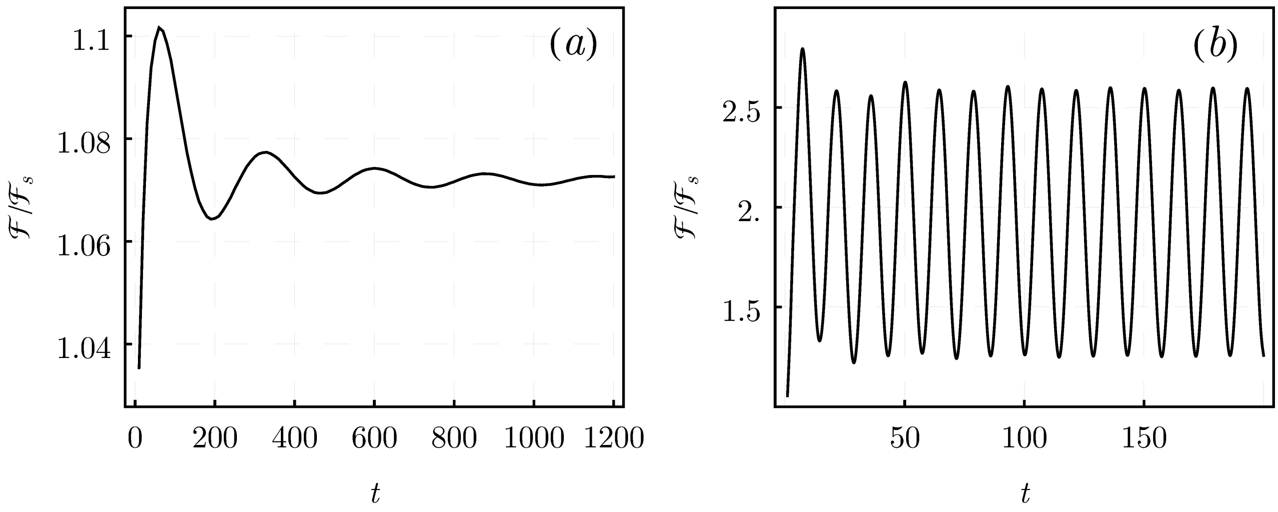


Figure 3.1: Ratio of the FCS $\mathcal{F}(\lambda, t)$ (3.21) to the large t asymptotic formula $\mathcal{F}_s(\lambda, t)$ given by (3.29), the initial state is characterized by $k_F = 1$, $E_F = k_F^2 = 1$, $\rho(E) = \theta(E_F - E)$: (a) delta barrier $V(x) = g\delta(x)$, $g = -0.3$ (one bound state), $\lambda = 0.3$; (b) symmetric double delta barrier potential (3.87) with $d = 2.3$, $g = -1.3$ (two bound states), $\lambda = 1.3$.

are absent in the spectrum or there is only one bound state then we expect only decaying transient time dependence of $C(\lambda, t) \approx C(\lambda)$, see figure 3.1(a). This way, in equation (3.28), we recover predictions for the FCS also known as the Levitov–Lesovik formula [152, 153, 154]. The large deviation theory perspective on this formula can be found in [162], while the generalized hydrodynamic point of view is presented in [163]. When the function $\rho(E)$ has sharp jumps, as it happens, for instance, at zero temperature $\rho(E) = \theta(E_F - E)$, or for the non-equilibrium setups [164, 165], then additionally to the smooth time dependence in $C(\lambda, t)$, we obtain also power law dependencies, with the corresponding exponents defined by the value of the function $\nu_\lambda(E)$ at the jump points. In particular, the modification of the vacuum case reads

$$\log \mathcal{F}_s(\lambda, t) = -\left(\nu_\lambda(0)^2 + \nu_\lambda(E_F)^2\right) \log t + \frac{t}{2\pi} \int_0^{E_F} \log(1 + (e^\lambda - 1)T(E))dE. \quad (3.29)$$

Notice that $\nu_\lambda(0) = 0$ for a generic barrier since $T(E = 0) = 0$. However for special potentials with $T(E = 0) \neq 0$ (e.g. reflectionless potentials) $\nu_\lambda(0) \neq 0$ also gives a contribution to (3.29).

Finally, when there are two or more bound states in the spectrum, then $C(\lambda, t)$ contains persistent oscillatory contributions with the frequency equal to the difference of energies of the bound states, see figure 3.1(b). Notice that after a few periods oscillations are described by one harmonic with a constant amplitude. For a specific defect in a lattice model this was demonstrated in [160].

3.3.1 Entanglement Entropy

Let us also mention that one can relate the entanglement entropy $\mathcal{S}(t)$ obtained after tracing out the left part of the system to the FCS by a simple integral [166, 167, 168, 169]. We express this relation in a simple and convenient form as

$$\mathcal{S}(t) = \frac{1}{4}\int\limits_{-\infty}^{\infty} \frac{\log \mathcal{F}(\lambda, t)}{\sinh^2(\lambda/2)} d\lambda, \tag{3.30}$$

where the integral at $\lambda = 0$ should be treated in the principal value sense. Substituting instead of complete $\mathcal{F}$ its asymptotic expression $\mathcal{F}_s$ for instance for zero temperature case (3.29), we obtain as $t \to \infty$

$$\mathcal{S}(t) \approx t \int\limits_{0}^{E_F} \frac{dE}{2\pi}\Big(- T(E)\log T(E) - R(E)\log R(E)\Big) - \frac{\log t}{4}\int\limits_{-\infty}^{\infty} \frac{\nu_\lambda(0)^2 + \nu_\lambda(E_F)^2}{\sinh^2(\lambda/2)} d\lambda, \tag{3.31}$$

Here $R(E) \equiv 1 - T(E)$. The linear in time part of this formula is generic for one-dimensional systems [170], and in this case it has a form of classical Shannon entropy (see also [171] and [172]), the suitable generalization to the interacting systems was obtained in [173]. The logarithmic growth becomes important in the case of the absence of the defect, or for the reflectionless potential, when the linear part disappears. The coefficient in front of the logarithm is compatible with predictions from conformal field theories [174, 175, 171]

$$\mathcal{S}(t) = \frac{c}{6}\log t + O(1), \quad t \to \infty. \tag{3.32}$$

In our case for $T(E) = 1$ we get $c = 2$ after computing the integral in the last line of (3.31). Notice that the coefficient in front of the logarithmic correction when the linear part is present can be non-universal similarly to [171].

3.4 Hard-wall wave functions

The key part in deriving explicit expression of kernels is an explicit presentation for the hard-wall wave functions (3.13), (3.14) in terms of the Jost functions and scattering data. We start with χ_k. Assuming that the range of the potential ξ is much smaller than R, the wave function can be presented as

$$\chi_k(x) = \mathrm{Im}\left[e^{ikR}\psi_k(x)\right], \tag{3.33}$$

where ψ_k is a Jost function that corresponds to the potential $V(x)$ (see (3.4)). This way the condition $\chi_k(R) = 0$ is satisfied automatically, while for the large negative x the behavior reads

$$\chi_k(x) = \mathrm{Im}\left[e^{ikR}(\bar{a}_k e^{-ikx} - b_k e^{ikx})\right]. \tag{3.34}$$

Here the scattering data corresponds to the potential $V(x)$. Demanding $\chi_k(-R) = 0$ will provide us with the spectrum condition, that can be resolved as

$$e^{2ikR} = \frac{i\mathrm{Im}\, b_k + \sqrt{1 + (\mathrm{Re}\, b_k)^2}}{\bar{a}_k} \equiv e^{-2i\delta(k)}. \tag{3.35}$$

Here we have introduced the scattering phase $\delta(k)$. We have to take into account two possible solutions that correspond to two different branches of the square root. This way, in fact we have two different scattering phases. For both of them we have $\delta(k) = -\delta(-k)$, meaning that if k is a solution than $-k$ is solution as well, with the same energy $E_k = k^2$. However, they describe the same state as is clearly seen from (3.33). Theqrefore, we restrict ourselves to the positive k solutions of (3.35).

Let us also discuss the normalization of the wave function. To this end we notice that the k derivative of the χ_k satisfies

$$\left(-\partial_x^2 + V(x) - k^2\right)\partial_k\chi_k = 2k\chi_k, \qquad \left(-\partial_x^2 + V(x) - k^2\right)\chi_k = 0. \tag{3.36}$$

So we can write

$$\begin{aligned}2k(\chi_k, \chi_k) &= \int\limits_{-R}^{R} dx \left[-\frac{d^2\partial_k\chi_k}{dx^2}\chi_k(x) + \partial_k\chi_k\frac{d^2\chi_k(x)}{dx^2}\right] \\ &= \left[-\frac{d\partial_k\chi_k}{dx}\chi_k(x) + \partial_k\chi_k\frac{d\chi_k(x)}{dx}\right]\Big|_{-R}^{R}.\end{aligned} \tag{3.37}$$

This allows us to present

$$(\chi_k, \chi_k) = (\mathrm{Re}\, b_k + \sqrt{1+(\mathrm{Re}\, b_k)^2})\sqrt{1+(\mathrm{Re}\, b_k)^2}(R+\delta'(k)). \tag{3.38}$$

Here $\delta'(k)$ means the momentum derivative. Similarly, we can describe the matrix elements $(\chi_k, P_> \chi_p) = \int\limits_0^R dx \chi_k(x)\chi_p(x)$ of the projector in (3.19) as

$$\begin{aligned}
&(E_k - E_p)(\chi_k, P_>\chi_p) = \\
&= \int\limits_0^R dx \left(\left[\left(-\partial_x^2 + V(x)\right) \chi_k(x) \right] \chi_p(x) - \chi_k(x) \left(-\partial_x^2 + V(x)\right) \chi_p(x) \right) \\
&\qquad = \int\limits_0^R dx \partial_x \left(-\chi_p(x)\partial_x \chi_k(x) + \chi_k(x)\partial_x \chi_p(x)\right) \\
&\qquad\qquad = \chi_p(0)\partial_x\chi_k(0) - \chi_k(0)\partial_x\chi_p(0).
\end{aligned} \tag{3.39}$$

To describe bound states that might be present in the system, one can argue that due to finite range of the potential the corresponding wave functions will be localized around $x = 0$, and decay exponentially for large x. Theqrefore the boundary conditions are satisfied automatically with the exponential precision, and we may put

$$\chi_k^{\text{bound}}(x) \approx \varphi_{i\varkappa}(x), \qquad k = i\varkappa. \tag{3.40}$$

Its normalization can be found in a similar manner taking into account the identification $\varphi_{i\varkappa}(x) = b_\varkappa \bar{\psi}_{i\varkappa}(x)$ discussed in Section 3.2. Indeed, using the fact that at $x \to +\infty$, the leading term in the momentum in the wave function behaves as $a'_{i\varkappa} e^{\varkappa x}$, we obtain

$$(\varphi_{i\varkappa}, \varphi_{i\varkappa}) = i a'_{i\varkappa} b_\varkappa. \tag{3.41}$$

Similarly we can find the pre-quench wave function Λ_q. In this case it is more convenient to use the Jost solution (3.5) on the potential V_0, which we denote as $\Phi_q(x)$. In this notation we propose the following formula

$$\Lambda_q(x) = \mathrm{Im}\frac{\Phi_q(x)}{\Phi_q(0)}. \tag{3.42}$$

Notice that in this form the boundary condition $\Lambda_q(0) = 0$ is satisfied automatically, while the condition $\Lambda_q(-R) = 0$ defines spectrum and the scattering phase

$\eta(q)$

$$e^{2iqR} = \frac{\Phi_q(0)}{\bar{\Phi}_q(0)} \equiv e^{-2i\eta(q)}. \tag{3.43}$$

Normalization now reads as

$$(\Lambda_q, \Lambda_q) = \frac{R + \eta'(q)}{2|\Phi_q(0)|^2}. \tag{3.44}$$

Finally, computation of the overlaps between pre- and post-quench wavefunctions in (3.19), can be avoided completely, and replaced by the corresponding overlaps with the Jost's functions. Namely, as it follows from (3.39) the time derivative of the (3.19) can be expressed via the (conjugated) time evolution of the wave function $\Lambda_q(y,t)$ defined as

$$\Lambda_q(y,t) \equiv \sum_k \frac{(\Lambda_q, \chi_k)\chi_k(y)}{(\chi_k, \chi_k)} e^{itE_k} = \int_{-R}^{0} dx \Lambda_q(x) G^*(x,y,t). \tag{3.45}$$

Here we have used the following presentation of the Green's function

$$G^*(x,y,t) \equiv \sum_k \frac{\chi_k(x)\chi_k(y)}{(\chi_k, \chi_k)} e^{itE_k}. \tag{3.46}$$

The summation is taken over all spectral points (3.35). We perform this summation explicitly in E with the genuine discrete degrees of freedom and take the thermodynamic limit only in the very end. The computation is straightforward but a bit tedious. However, the obtained result can be easily explained heuristically. Namely, one can argue that in the thermodynamic limit instead of function (3.46) one can use (3.12). This way, we can find a presentation only with the Jost solutions introduced in the previous section

$$\Lambda_q(y,t) = \int_C \frac{dk}{2\pi} \frac{(\Lambda_q, \varphi_k)\bar{\psi}_k(y)}{a_k} e^{itE_k}. \tag{3.47}$$

The integration path C runs from $-\infty$ to $+\infty$ in the upper half plane above all positions of zeroes of a_k, see figure 3.2. The overlap (Λ_q, φ_k) can be computed using the same trick as in (3.37) and (3.39). Indeed, if we introduce function

$$\Xi_{q,k} = \Lambda_q'(0)\varphi_k(0) - \int_{-\infty}^{0} dx \Lambda_q(x)(V_0(x) - V(x))\varphi_k(x), \tag{3.48}$$

we can present

$$(E_k - E_q) \int\limits_{-R}^{0} dx \Lambda_q(x) \varphi_k(x) = \Xi_{q,k} - \Lambda_q'(-R)\varphi_k(-R). \tag{3.49}$$

Here we have used that due to the finite range of the potentials the lower limit of the integration in (3.48) can be either $-R$ or $-\infty$. Taking into account that for $k \in C$ the last term vanishes exponentially $\varphi_k(-R) \sim e^{ikR}$, we finally present

$$\Lambda_q(y,t) = \int_C \frac{dk}{2\pi} \frac{\Xi_{q,k} \bar{\psi}_k(y)}{(k^2 - q^2) a_k} e^{itE_k}. \tag{3.50}$$

This is the final answer in the thermodynamic limit. Notice that $\Xi_{q,k}$ is a regular function and can be continued from the discrete spectrum to upper half plane of the variable k. In the next section we will evaluate large-time asymptotic behavior of the kernel, which is mostly defined by $\Xi_{q,-q}$. It can be computed from (3.49) along with the asymptotic behavior $\Lambda_q'(-R) \sim -qe^{iqR}/\Phi_q(0)$ for large R (see (3.42))

$$\Xi_{q,-q} = -\frac{q}{\Phi_q(0)}. \tag{3.51}$$

This expression can be directly obtained from the definition (3.48) already in the thermodynamic limit. We demonstrate it in G. The direct computation of $\Lambda_q(0,t)$ and its derivative in the finite system is given in F.

3.5 Kernel

To compute the kernel $K(q,q')$ for the Fredholm determinant of the FCS (3.21), we start by considering its time derivative. Using explicit presentation (3.19) and (3.39), along with the definition (3.45), we arrive at

$$\frac{dK(q,q')}{dt} = \frac{2i(e^\lambda - 1)}{\pi} |\Phi_q(0)| \left(f_q^{(1)}(t) \bar{f}_{q'}^{(0)}(t) - f_q^{(0)}(t) \bar{f}_{q'}^{(1)}(t) \right) |\Phi_{q'}(0)|, \tag{3.52}$$

where we have denoted

$$f_q^{(\alpha)}(t) = \partial_x^\alpha \Lambda_q(x,t)\Big|_{x=0} = \int\limits_C \frac{dk}{2\pi} \frac{\Xi_{q,k} \partial_x^\alpha \bar{\psi}_k(0)}{a_k} \frac{e^{itk^2}}{k^2 - q^2}, \qquad \alpha = 0, 1. \tag{3.53}$$

The contour C runs as is shown in figure 3.2. Using presentation (3.50) we can

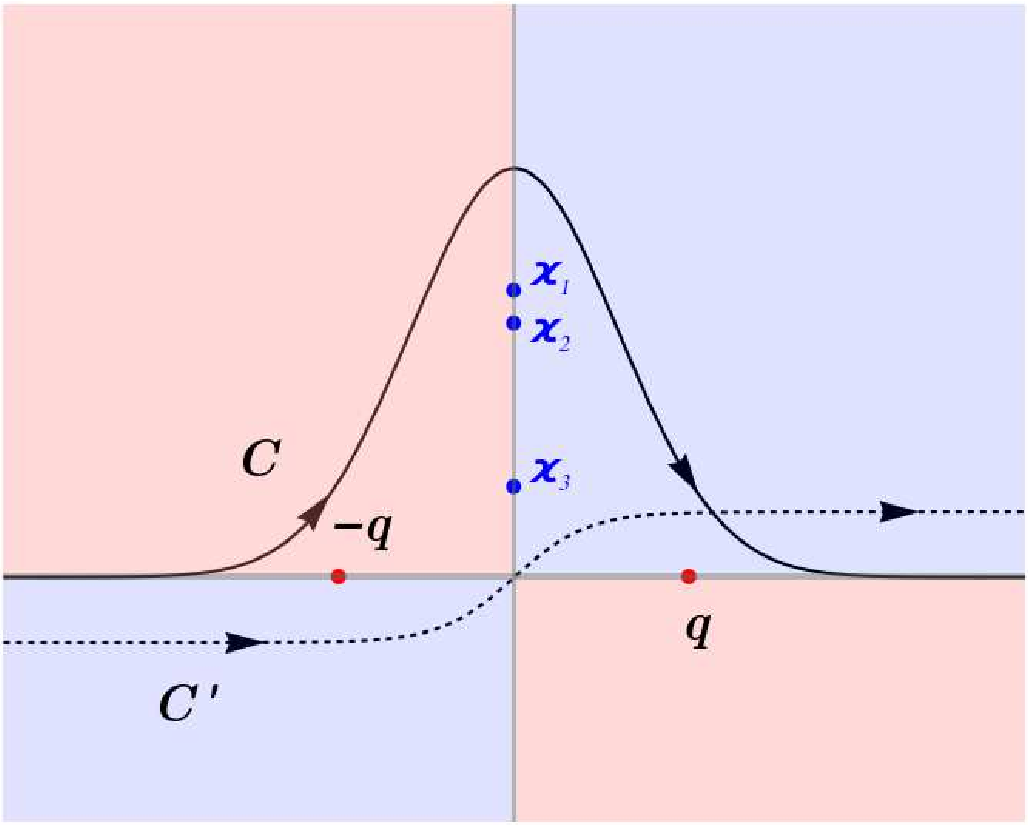


Figure 3.2: Integration contours C and C' in the complex plane of k for the integral presentation of $f_q^{(\alpha)}$ given by (3.53). The contours C are C' are the initial and transformed contours of integration, respectively. Blue dots on the imaginary axis correspond to the bound states, red dots correspond to poles at $k = \pm q$ in (3.53). The shaded areas show the regions of exponential decaying (I, III quadrants, light blue) and exponential growth (II, IV quadrants, pink) of $\exp(itq^2)$ for $t \to +\infty$.

directly integrate (3.52). However, in order to easier assess the long-time asymptotic behavior we first identically transform $f_q^{(\alpha)}$ to highlight the most relevant terms as $t \to +\infty$. To do so we notice that the exponential e^{itk^2} is decaying in the first and third quadrants of complex plane of k (see figure 3.2). So we deform the contour C into C' by pulling it towards the real negative line and crossing it. By doing so we inevitably encircle all positions of the bound states and the pole $k = -q$. The obtained deformation reads

$$f_q^{(\alpha)}(t) = \frac{i\Xi_{q,-q}\partial_x^\alpha \bar{\psi}_{-q}(0)}{a_{-q}} \frac{e^{itq^2}}{2q} + \sum_{n=1}^{N^{\mathrm{b}}} \frac{i\Xi_{q,i\varkappa_n}\partial_x^\alpha \bar{\psi}_{i\varkappa_n}(0)}{a'_{i\varkappa_n}} \frac{e^{-it\varkappa_n^2}}{\varkappa_n^2 + q^2} + \int\limits_{C'} \frac{dk}{2\pi} \frac{\Xi_{q,k}\partial_x^\alpha \bar{\psi}_k(0)}{a_k} \frac{e^{itk^2}}{k^2 - q^2}. \tag{3.54}$$

The “leading” coefficient $\Xi_{q,-q}$ was computed in (3.51). Further we use the symmetry $k \to -k$ to fold the full contour C' and consider integration only with $\operatorname{Re} k > 0$, namely

$$f_q^{(\alpha)}(t) = \sum_{n=1}^{N^{\mathrm{b}}} B_{n,q}^{(\alpha)} e^{-it\varkappa_n^2} + F_q^{(\alpha)} e^{itq^2} + \int\limits_0^\infty \frac{dk}{\pi} \Omega_{q,k}^{(\alpha)} \frac{e^{itk^2}}{(k+i0)^2 - q^2}, \tag{3.55}$$

$$B_{n,q}^{(\alpha)} = \frac{i\Xi_{q,i\varkappa_n}\partial_x^\alpha \bar{\psi}_{i\varkappa_n}(0)}{a'_{i\varkappa_n}(\varkappa_n^2 + q^2)}, \qquad F_q^{(\alpha)} = -i\frac{\partial_x^\alpha \psi_q(0)}{2\Phi_q(0)a_{-q}}, \tag{3.56}$$

$$\Omega_{q,k}^{(\alpha)} = \operatorname{Re} \frac{\Xi_{q,k}\partial_x^\alpha \bar{\psi}_k(0)}{a_k}. \tag{3.57}$$

Such form of $f_q^{(\alpha)}(t)$ is convenient for large t asymptotic analysis. The first two terms give persistent oscillations, while the integral in (3.55) is decaying as a power law in t for large t. This can be deduced from the stationary phase method considering a saddle point at $k = 0$. The corresponding exponent of the power law decay depends on the behavior of $\Omega_{q,k}^{(\alpha)}$ at $k = 0$. In the case of generic potentials, a_k has a first order pole at $k = 0$ while $\Xi_{q,k}$ and $\partial_x^\alpha \psi_k(0)$ are regular at $k = 0$. Theqrefore $\Omega_{q,k}^{(\alpha)}$ has at least first order zero at $k = 0$, which implies the entire integral to be estimated as $O(t^{-1})$. For some special potentials (for example reflectionless potentials), a_k is regular at $k = 0$. For such potentials the integral decays as $t^{-1/2}$

$$\int_0^\infty \frac{dk}{\pi}\Omega^{(\alpha)}_{q,k}\frac{e^{itk^2}}{(k+i0)^2-q^2}=\frac{I_q^{(\alpha)}}{\sqrt{t}}+O(t^{-1}), \tag{3.58}$$

$$I_q^{(\alpha)}=-\frac{\sqrt{\pi}e^{i\pi/4}\Xi_{q,0}\partial_x^\alpha\psi_0(0)}{2a_0q^2}. \tag{3.59}$$

To compute the kernel we substitute $f^{(\alpha)}(t)$ in the form (3.55) into (3.52) and integrate over t. Additionally, we perform conjugation with diagonal matrices

$$K(q,q')\to K(q,q')e^{-it(E_q-E_{q'})/2}. \tag{3.60}$$

This operation does not change the determinant, so for the transformed kernel we obtain

$$K(q,q')=K_0(q,q')+\delta K(q,q'). \tag{3.61}$$

Here $K_0(q,q')$ is given by

$$\begin{aligned}K_0(q,q')=&\frac{4i(e^\lambda-1)}{\pi}\\&\times|\Phi_q(0)|(F_q^{(1)}\bar{F}_{q'}^{(0)}-F_q^{(0)}\bar{F}_{q'}^{(1)})|\Phi_{q'}(0)|\frac{\sin t(E_q-E_{q'})/2}{E_q-E_{q'}}.\end{aligned} \tag{3.62}$$

Using definition (3.56) it can be equivalently presented as (3.24). The rest of the kernel can be presented as

$$\delta K(q,q')=\frac{2i(e^\lambda-1)}{\pi}|\Phi_q(0)|\left(M_{qq'}(t)-\bar{M}_{q'q}(t)\right)|\Phi_{q'}(0)| \tag{3.63}$$

with

$$M_{qq'}(t)=e^{-it(E_q-E_{q'})/2}\sum_{i=1}^4\left[K^{(i)}(q,q',t)-K^{(i)}(q,q',0)\right]. \tag{3.64}$$

Here different kernels have different physical meaning. The kernel $K^{(1)}$ is responsible for the contribution of the bound states only. It is given by

$$K^{(1)}(q,q',t)=\sum_{m<n}^{N^{\mathrm{b}}}(B^{(1)}_{mq}B^{(0)}_{nq'}-B^{(0)}_{mq}B^{(1)}_{nq'})\frac{e^{it(\varkappa_n^2-\varkappa_m^2)}}{i(\varkappa_n^2-\varkappa_m^2)}. \tag{3.65}$$

The kernel $K^{(2)}$ is responsible for contribution of the continuous spectrum only

$$K^{(2)}(q,q',t) = \int_0^\infty \frac{dk}{\pi} \frac{e^{it(E_k - E_{q'})}}{i(E_k^+ - E_{q'})} \frac{\Omega_{qk}^{(1)} \bar{F}_{q'}^{(0)} - \Omega_{qk}^{(0)} \bar{F}_{q'}^{(1)}}{E_k^+ - E_q} + \frac{1}{2} \int\limits_0^\infty \frac{dk}{\pi} \int\limits_0^\infty \frac{dp}{\pi} \frac{e^{it(E_k - E_p)}}{i(E_k^+ - E_p^-)} \frac{\Omega_{qk}^{(1)} \Omega_{q'p}^{(0)} - \Omega_{qk}^{(0)} \Omega_{q'p}^{(1)}}{(E_k^+ - E_q)(E_p^- - E_{q'})}, \quad (3.66)$$

here $E_k = k^2$ and $E_k^\pm = (k \pm i0)^2$. Finally the kernels $K^{(3)}$ and $K^{(4)}$ give the mixed contribution from the bound states and the continuous spectrum

$$K^{(3)}(q,q',t) = \sum_{n=1}^{N^{\mathrm{b}}} \int_0^\infty \frac{dk}{\pi} \frac{e^{it(E_k + \varkappa_n^2)}}{i(E_k^+ + \varkappa_n^2)} \frac{\Omega_{qk}^{(1)} B_{nq'}^{(0)} - \Omega_{qk}^{(0)} B_{nq'}^{(1)}}{E_k^+ - E_q}, \quad (3.67)$$

$$K^{(4)}(q,q',t) = \sum_{n=1}^{N^{\mathrm{b}}} (B_{nq'}^{(0)} F_q^{(1)} - B_{nq'}^{(1)} F_q^{(0)}) \frac{e^{it(\varkappa_n^2 + E_q)}}{i(E_q + \varkappa_n^2)}. \quad (3.68)$$

Integrals in $K^{(2)}$ and $K^{(3)}$ decay for large t because of averaging of rapid oscillations as in the integral (3.55). Special care has to be taken for the reflectionless potentials. At the first glance, in this case relations (3.58), (3.59) might produce a logarithmic growth for large t in the double integral in $K^{(2)}$. This growth is, however, absent because of the relation

$$I_q^{(1)} \bar{I}_{q'}^{(0)} - I_q^{(0)} \bar{I}_{q'}^{(1)} = 0. \quad (3.69)$$

There are also potential singularities for small $q \lesssim t^{-1/2}$ and a bit different asymptotic analysis of (3.58) is needed. Indeed, (3.59) shows a singular behavior for small q, which in fact is not there, since in the asymptotic analysis of (3.58) we have assumed that a pole at $k = q$ is far from the stationary point $k = 0$. We performed such analysis for the current and showed that the contribution of small q gives only the subleading contributions.

Apart from the decaying terms, δK contains also time-independent terms $K^{(i)}(t = 0)$, highly oscillating terms like $K^{(4)}(t)$, and terms that oscillate with the frequencies given by the energies of the bound states $K^{(1)}(t)$. The latter comes in the form of the finite rank operators, and can appear in the final expression of the determinant only linearly. As we have discussed in Section 3.3 we expect that the contribution of the kernel δK to the asymptotic analysis of the Fredholm operator $\det(1+\hat{K})$ enters only as a smooth overall prefactor, which has non-vanishing time dependence only if there are two or more bound states in the spectrum.

3.5.1 FCS for perfect lead attachment

There are well-developed methods for asymptotic analysis of the Fredholm determinants of the so-called integrable kernels [94, 176]. As we have shown above for generic potentials $V_0(x)$ and $V(x)$ the kernel for FCS $K(q,q')$ is not an integrable one.

In this subsection we consider a special case of quench setup when the obtained kernel is integrable even for finite times. We call this situation the *perfect lead attachment* because it corresponds to the scenario when $V_0(x) = V(x)$ for $x < 0$.

In this case due to the integral presentation (3.5) the corresponding Jost functions coincide for negative x: $\varphi_q(x) = \Phi_q(x)$ for $x \le 0$. From presentation (3.48) we observe the factorization

$$\Xi_{q,k} = \Lambda_q'(0)\varphi_k(0), \tag{3.70}$$

which imply a similar factorization $f_q^{(\alpha)}(t) = \Lambda_q'(0) g_q^{(\alpha)}(t)$ for $f_q^{(\alpha)}(t)$ given by (3.53), where

$$g_q^{(\alpha)}(t) = \int\limits_C \frac{dk}{2\pi}\omega_k^{(\alpha)}\frac{e^{itk^2}}{k^2-q^2}, \qquad \omega_k^{(\alpha)} \equiv \frac{\varphi_k(0)\partial_x^\alpha\bar{\psi}_k(0)}{a_k}. \tag{3.71}$$

Comparing (3.70) at $k=-q$ with (3.51) we conclude that $\Lambda_q'(0) = -q/|\varphi_q(0)|^2$. Theqrefore now (3.52) reads

$$\frac{dK(q,q')}{dt} = \frac{2i(e^\lambda-1)qq'}{\pi|\varphi_q(0)||\varphi_{q'}(0)|}\left(g_q^{(1)}(t)\bar{g}_{q'}^{(0)}(t) - g_q^{(0)}(t)\bar{g}_{q'}^{(1)}(t)\right). \tag{3.72}$$

Integrating in t we can present the kernel $K(q,q')$ in the integrable form

$$K(q,q') = \frac{2(e^\lambda-1)qq'}{\pi|\varphi_q(0)||\varphi_{q'}(0)|} \times \frac{g_q^{(1)}(t)\bar{g}_{q'}^{(0)}(t) - g_q^{(0)}(t)\bar{g}_{q'}^{(1)}(t) + \bar{D}_q(t) - D_{q'}(t)}{E_q - E_{q'}}, \tag{3.73}$$

where

$$D_q(t) = i\int\limits_0^t d\tau \int\limits_C \frac{dk}{2\pi} e^{i\tau k^2}\left[\omega_k^{(1)}\bar{g}_q^{(0)}(\tau) - \omega_k^{(0)}\bar{g}_q^{(1)}(\tau)\right]. \tag{3.74}$$

To check correctness of (3.73) we need to compare its derivative in t with (3.72) using

$$\frac{d}{dt}g_q^{(\alpha)}(t) = iq^2 g_q^{(\alpha)}(t) + i\int\limits_C \frac{dk}{2\pi}\omega_k^{(\alpha)}e^{itk^2}. \tag{3.75}$$

Also we have to check that $K(q,q') = 0$ at $t = 0$. This is ensured due to the property $g_q^{(0)}(0) = 0$, which follows from analyticity of $\omega_k^{(0)}$ in the upper half-plane of k. The integrable form of kernel $K(q,q')$ allows one to replace evaluation of the Fredholm determinants by a solution of the Riemann–Hilbert problem [94, 176]. This approach is especially useful for the asymptotic analysis at large time $t \to +\infty$. In this case, however, if we follow the standard procedure outlined in [176], the corresponding jump matrix will have size 4×4. Theqrefore, we postpone full analysis to a separate publication.

The asymptotic behavior of $g_q^{(\alpha)}(t)$ can be found similarly to (3.55), where one can neglect the last integral. To find the large-time asymptotic behavior of $D_q(t)$ we present it identically as

$$\begin{aligned} D_q(t) &= \int\limits_C \frac{dk}{2\pi}\int\limits_{C^*}\frac{dp}{2\pi}\frac{e^{it(k^2-p^2)}-1}{k^2-p^2}\frac{\bar{\omega}_p^{(0)}\omega_k^{(1)} - \bar{\omega}_p^{(1)}\omega_k^{(0)}}{p^2-q^2} \\ &\approx -\int\limits_C \frac{dk}{2\pi}\int\limits_{C^*}\frac{dp}{2\pi}\frac{1}{k^2-p^2+i0}\frac{\bar{\omega}_p^{(0)}\omega_k^{(1)} - \bar{\omega}_p^{(1)}\omega_k^{(0)}}{p^2-q^2}. \end{aligned} \tag{3.76}$$

Here C^* is a contour conjugated to C. Moreover, for the symmetric potential function $g_q^{(1)}(t)$ simplifies significantly and the integral can be dropped even for finite times, namely, we can present

$$g_q^{(1)}(t) = \frac{e^{itq^2}}{2\bar{a}_q}. \tag{3.77}$$

Here we used that for arbitrary even potential $V(-x) = V(x)$, the Jost solutions are related as $\psi_{-k}(x) = \varphi_k(-x)$, which leads to

$$\omega_k^{(1)} = \frac{\varphi_k(0)\partial_x\psi_{-k}(0)}{a_k} = ik. \tag{3.78}$$

Indeed taking into account that the Wronskian $\varphi_k(x)\partial_x\psi_{-k}(x) - \psi_{-k}(x)\partial_x\varphi_k(x)$ does not depend on x and calculating it at $x \to -\infty$ and $x = 0$ we obtain the relation (3.78). Thus, the integral in (3.55) vanishes identically, since it depend only on the real part of (3.78). Further the bound state contribution vanishes because the wave-functions are either odd or even, meaning that either the value at zero or the value of the derivative at zero vanishes leading to $\varphi_{i\varkappa_n}(0)\partial_x\bar{\psi}_{i\varkappa_n}(0) = 0$.

3.6 The current

Let us also discuss the full current $J(t)$ of the particles flowing through the middle $x = 0$ to the right part of the system. It can be evaluated from the FCS (3.21) as follows

$$
\begin{aligned}
J(t) = \frac{d}{dt}\frac{d\mathcal{F}(\lambda,t)}{d\lambda}\Big|_{\lambda=0} &= \mathrm{Tr}\left(\rho\frac{d}{dt}\frac{d\hat{K}}{d\lambda}\Big|_{\lambda=0}\right) \\
&= -\int_0^\infty dq\rho(q)\frac{4|\Phi_q(0)|^2}{\pi}\mathrm{Im}\, f_q^{(1)}(t)\bar{f}_q^{(0)}(t), \quad (3.79)
\end{aligned}
$$

where at the last step we used explicit presentation (3.52) to compute the trace. As we discuss in Section 3.5, the integral in (3.55) may be dropped for the calculation of current for large t since it vanishes as $t \to \infty$, and we can approximate

$$f_q^{(\alpha)}(t) \approx F_q^{(\alpha)}e^{itq^2} + \sum_{n=1}^{N^{\mathrm{b}}} B_{n,q}^{(\alpha)}e^{-it\varkappa_n^2}. \quad (3.80)$$

Substituting this expression into (3.79) we obtain three type of contributions to the current

$$J(t) \approx J_{\mathrm{LB}} + J^{\mathrm{b}} + \delta J, \quad (3.81)$$

where J_{LB} comes from the first term in (3.80), J^{b} comes from the terms that involve the bound states only and δJ described the mix of the first term with the bound states.

To calculate J_{LB} we use $\mathrm{Im}\,\psi_q'(0)\bar{\psi}_q(0) = -q$ and (3.9)

$$J_{\mathrm{LB}} = \int_0^\infty \frac{dq}{\pi}\frac{q\rho(q)}{|a_q|^2} = \int \frac{dE}{2\pi}\rho(E)T(E). \quad (3.82)$$

It is well-known Landauer–Büttiker formula for the current.

The contribution of bound states to the current is

$$J^{\mathrm{b}} = \sum_{m<n} A_{mn} \sin t(\varkappa_m^2 - \varkappa_n^2), \tag{3.83}$$

where

$$A_{mn} = \frac{4\left(\bar{\psi}'_{i\varkappa_n}(0)\bar{\psi}_{i\varkappa_m}(0) - \bar{\psi}'_{i\varkappa_m}(0)\bar{\psi}_{i\varkappa_n}(0)\right)}{a'_{i\varkappa_m} a'_{i\varkappa_n}} \times \int_0^\infty \frac{dq}{\pi}\rho(q)|\Phi_q(0)|^2 \frac{\Xi_{q,i\varkappa_m}\Xi_{q,i\varkappa_n}}{(\varkappa_m^2 + q^2)(\varkappa_n^2 + q^2)}. \tag{3.84}$$

For the symmetric potential $V(x)$, the bound states are either even functions with $\bar{\psi}'_{i\varkappa_n}(0) = 0$ or odd functions with $\bar{\psi}_{i\varkappa_n}(0) = 0$. Theqrefore, in this case, a nontrivial contribution to the current may arise only from pairs of odd-even states. Furthermore, in the case of perfect lead attachment, $V(x) = V_0(x)$, we have $\Xi_{q,i\varkappa_n} = 0$ for odd bound states $\bar{\psi}_{i\varkappa_n}(x)$ and theqrefore there is no contribution at all to the current from bound states in the case of perfect lead attachment with an even potential.

The integral in q for δJ can be estimated by the contribution at $q = 0$ by the method of stationary phase and it can be shown that δJ decays for large t at least as $t^{-1/2}$ and theqrefore does not give a leading contribution to the current.

Finally we arrive to the following expression for the large-time asymptotic current

$$J(t) \approx \int \frac{dE}{2\pi}\rho(E)T(E) + \sum_{m<n} A_{mn} \sin t(\varkappa_m^2 - \varkappa_n^2). \tag{3.85}$$

We see that in addition to the constant Landauer–Büttiker current (the first term), there are also oscillating terms connected with the presence of the multiple bound states.

To illustrate this formula we consider an example of the reflecionless potential $V(x) = -2/\cosh^2 x$. For this potential $T(E) = 1$, hence the name. The Jost functions and functions $f_q^{(\alpha)}(t)$ can be easily computed and the results are

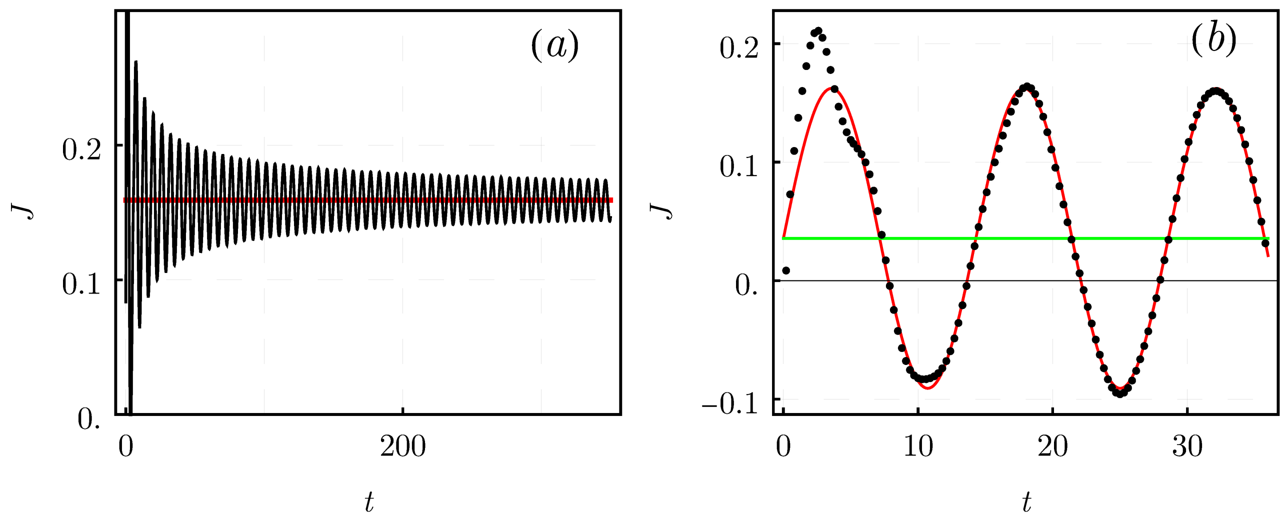


Figure 3.3: Current through the point $x = 0$ and its asymptotic behavior, the initial state is characterized by $k_F = 1$, $E_F = k_F^2 = 1$, $\rho(E) = \theta(E_F - E)$: (a) the reflectionless potential $V(x) = -2/\cosh^2 x$ (one bound state); the current (black) is oscillating with an amplitude decaying as $\sim t^{-1/2}$ around Landauer–Büttiker constant current $J_{LB} = E_F/(2\pi)$ (red). (b) symmetric double delta barrier potential (3.87) with $d = 2.3$, $g = -1.3$ (two bound states); the current (black dots) has asymptotic oscillating behavior (3.90) with fixed amplitude (red curve) around Landauer–Büttiker constant current (green line).

presented in H.2. The exact expression for the current than reads as (3.79)

$$J(t) = \int_0^\infty \frac{dq}{\pi}\rho(q) \times \left(q + \sin[(1+q^2)t] + 2(1+q^2)\mathrm{Im} \int\limits_0^\infty \frac{dk}{\pi} \frac{k^2}{1+k^2} \frac{e^{it(k^2-q^2)}}{(k+i0)^2 - q^2} \right). \quad (3.86)$$

We plot this expression for $\rho(E) = \theta(E_F - E)$ in figure 3.3(a) against the Landauer–Büttiker expression $J_{LB} = E_F/(2\pi)$. Notice that even though the bound state is present in the spectrum, it produces only vanishing with time oscillations.

To demonstrate the persistent oscillations we consider the symmetric double delta barrier potential

$$V(x) = g\delta(x - d/2) + g\delta(x + d/2). \quad (3.87)$$

The corresponding scattering data can be computed explicitly (for the details see H.3)

$$a_k = \frac{g^2 e^{2ikd} + (2k + ig)^2}{4k^2}, \quad b_k = \frac{ge^{-idk}(g - 2ik) - ge^{idk}(g + 2ik)}{4k^2}. \quad (3.88)$$

The bound states momenta follow from the relation $a_{i\varkappa} = 0$, which if we define $u = 2\varkappa/|g|$, $D = |g|d$ can be written as

$$(u-1)^2 - e^{-uD} = 0. \quad (3.89)$$

For the negative couplings this equation has two solutions for $D > 2$ and one solution for $0 \le D \le 2$. Note a_k has a simple pole at $k = 0$ if $D \ne 2$. The case $D = 2$ describes a situation when the bound states is just starts to appear from (disappear into) the continuous spectrum, which formally is reflected in a_k being regular at $k = 0$. Notice that same behavior is inherent for the reflectionless potentials, while for generic potentials a_k has a simple pole at $k = 0$. The formula for the asymptotic current (3.85) is now given by

$$J(t) \approx \int \frac{dE}{2\pi}\rho(E)T(E) + A_{12} \sin t(E_2 - E_1), \quad (3.90)$$

where $T(E) = |a_k|^{-2}$ is the transmission coefficient; the energies of bound states $E_j = -\varkappa_j^2$ are defined via solutions $\varkappa_j$ of the equation (3.89); the amplitude

A_{12} follows from (3.84) and is presented explicitly in (H.52). In figure 3.3(b) we compare the asymptotic current (3.90) with the exact expression (3.79) computed numerically using $f_q^{(\alpha)}(t)$ given in H.3. We observe that the asymptotic regime establishes after few oscillations.

3.7 Conclusions

To summarize, we have presented derivations of the Full Counting Statistics for the one-dimensional transport via an arbitrary defect from the first principles. The derivation in the main part is based on the effective presentation of the Green's function in the thermodynamic limit. The procedure of taking this limit (replacing the summation of the quantized quasimomenta to the integral) is not absolutely rigorous, so in the appendix, we have presented an exact summation over the quantized momenta with the subsequent thermodynamic limit. The final answer can be expressed via the Fredholm determinant whose numerical evaluation is straightforward.

We speculate that the large-time asymptotic behavior of the obtained Fredholm determinant could be deduced after certain approximations of the kernels, which render the determinant to be of the sine-kernel type. In this form, the answer depends only on the transmission coefficient of the post-quench potential, while the correlations of the original state are present only as the energy distribution. After these approximations, the Fredholm determinant could be analyzed either by the non-linear steepest descent method for the corresponding Riemann–Hilbert problem or by application the effective form factors. This way we were able to recover the Levitov–Lesovik formula and its modification by logarithmic corrections in case of discontinuous initial distributions.

As for the future directions, one can turn to the special quench of the *perfect lead attachment* when the obtained exact kernel is an integrable one and the Riemann–Hilbert problem appears without any approximations (see Section 3.5.1). It would be also interesting to develop effective form factor methods to find large-time asymptotic behavior directly from the series (3.17). Besides, these methods could be used to describe the situation when the Levitov–Lesovik formula is not applicable, i.e. when there are two or more bound states present in the spectrum of the post-quench potential and the FCS gets persistent oscillating behavior even for the constant potential bias. We plan to clarify how the amplitudes of these oscillations depend on the initial conditions and whether some memory effects of the pre-quench potentials are present.

In this chapter, we have not considered the case when there are bound states present in the pre-quench potential, but this case can be easily addressed in our formalism. Much more involved improvement of the formalism would be needed to tackle more general initial states (in particular, when there are some particles on the right-hand side of the system $\langle N_R(0)\rangle \neq 0$); to describe spinful electron and superconducting setups, and to explore the case of the driven system i.e. when the defect depends on time (for example, for the harmonically driven conformal defect [177]).

Apart from introducing the spin degrees of freedom, it would be interesting to address the multichannel scenario, which is more relevant to the theoretical description of the mesoscopic experiments. Indeed, in the typical setup, the leads are infinite in only one dimension while confinement in other dimensions creates additional channels connected with the possibility to excite transverse modes [178, 179]. We expect that the corresponding Fredholm determinants for the Full Counting Statistics will contain block kernels as the transmission coefficient $T(E)$ will become a matrix.

It is worth noting that our main assumption is based on the validity of the description of the electrons as essentially non-interacting fermions. This assumption is valid for equilibrium situations as a virtue of the Landau–Fermi theory and might be violated for non-equilibrium situations as we have here. We expect however that it remains valid as for the typical descriptions of the transport in driven nanoscale systems [180]. Physically, we require the existence of the quasiparticles with a lifetime sufficient for the proposed effects to be detected. In our case, this has to be larger than the frequency defined by the energy differences of two bound states.

Finally, let us mention that the experiments with ultracold atoms open a new venue to study transport in truly one-dimensional systems [181]. There the interactions are taken into account within the bosonization theory. We expect that the results of bosonization could be seen in the asymptotics of the corresponding

Fredholm determinants (as it was for free fermions [160]). However, the complete inclusion of interaction in the leads requires a separate investigation. There is also a full analog of the Landauer–Büttiker formalism for the interaction on the defect [182]. It would marvelous to find analogous formulas for the FCS, which is very challenging with our formalism.

Conclusions

In **chapter 1**, the effective form factors were introduced to simulate the static correlation functions for the states with finite entropy. The state was determined by the phase shift function $\nu(q)$. For the traditional approaches dealing with the finite entropy states is notoriously difficult but for our approach it is rather advantageous situation, since almost all available quantum numbers are occupied which tremendously simplifies the computation of form factor series. This allowed us, in particular, to re-derive known asymptotics for the static two point correlators in the XY spin chain and present them in a more compact form.

Additionally, it was established a relation between two representations of the correlation function, namelly, the first one is in the form of a difference of two Fredholm determinants and another one is in the form of a single Fredholm determinant. This fact allowed us to relate the tau function with arbitrary integer winding number δ to the problem of the asymptotic analysis of Toeplitz determinants with the winding number $\delta - 1$ and re-derive the Hartwig–Fisher theorem.

In **chapter 2**, it was found the asymptotics of dynamical correlation functions of anyonic gas with the parameter of anyonic statistics $0 \leq \kappa < 1$ using the effective form-factor approach. The main difficulty of this method is to find the phase-shift function $\nu(q)$ for effective fermions solving an integral equation. For large x and t we found approximate solutions for this integral equation that depend on the ratio $v = x/t$.

For the space-like region, $v > 1$, the solution $\nu(q)$ can be approximated by the smooth function $\nu_+(q)$. In this case, the asymptotics of the correlation function is given by asymptotic analysis of integrals producing the leading contribution either from a pole or from a saddle point. In the case of saddle-point contribution there is an additional power factor correcting the exponential decay of the correlation function.

For the time-like region, $|v| < 1$, we approximate the solution $\nu(q)$ for a large finite t by a function having discontinuities at critical points and corresponding to the solution of the integral equation at $t = \infty$. For large finite t we consider a class of regularized $\nu(q)$ having the same limit at $t = \infty$ as the genuine solution. It is remarkable, that the regularized $\nu(q)$ lead to the same asymptotics up to a prefactor independent of t. This universal time dependence of asymptotics has an additional power-like factor to the exponential decay of the correlation function. The exponent of this power-like factor is related directly to the jumps of $\nu(q)$ at critical points.

In **chapter 3**, we have presented derivations of the Full Counting Statistics for the one-dimensional transport via an arbitrary defect from the first principles. The derivation in the main part is based on the effective presentation of the Green's function in the thermodynamic limit. The final answer can be expressed via the Fredholm determinant whose numerical evaluation is straightforward.

It was argued that the large-time asymptotic behavior of the obtained Fredholm determinant could be deduced after certain approximations of the kernels, which render the determinant to be of the sine-kernel type. In this form, the answer depends only on the transmission coefficient of the post-quench potential, while the correlations of the original state are present only as the energy distribution. After these approximations, the Fredholm determinant could be analyzed either by the non-linear steepest descent method for the corresponding Riemann–Hilbert problem or by application of the effective form factors. This way we were able to recover the Levitov–Lesovik formula and its modification by logarithmic corrections in case of discontinuous initial distributions (e.g. in zero temperature case).

References

[1] O. Gamayun, N. Iorgov and Y. Zhuravlev, *Effective free-fermionic form factors and the XY spin chain*, SciPost Physics **10**(3) (2021), doi:10.21468/scipostphys.10.3.070.

[2] Y. Zhuravlev, E. Naichuk, N. Iorgov and O. Gamayun, *Large-time and long-distance asymptotics of the thermal correlators of the impenetrable anyonic lattice gas*, Physical Review B **105**(8) (2022), doi:10.1103/physrevb.105.085145.

[3] O. Gamayun, Y. Zhuravlev and N. Iorgov, *On Landauer–Büttiker formalism from a quantum quench*, Journal of Physics A: Mathematical and Theoretical **56**(20), 205203 (2023), doi:10.1088/1751-8121/accabf.

[4] M. Gaudin and J.-S. Caux, *The Bethe Wavefunction*, Cambridge University Press, doi:10.1017/cbo9781107053885 (2009).

[5] M. Takahashi, *Thermodynamics of One-Dimensional Solvable Models*, Cambridge University Press, doi:10.1017/cbo9780511524332 (1999).

[6] N. A. Slavnov, *Calculation of scalar products of wave functions and form factors in the framework of the alcebraic Bethe ansatz*, Theoretical and Mathematical Physics **79**(2), 502 (1989), doi:10.1007/bf01016531.

[7] N. A. Slavnov, *The algebraic Bethe ansatz and quantum integrable systems*, Russian Mathematical Surveys **62**(4), 727 (2007), doi:10.1070/rm2007v062n04abeh004430.

[8] S. Pakuliak, E. Ragoucy and N. Slavnov, *Nested Algebraic Bethe Ansatz in integrable models: recent results*, SciPost Physics Lecture Notes (2018), doi:10.21468/scipostphyslectnotes.6.

[9] N. Slavnov, *Introduction to the nested algebraic Bethe ansatz*, SciPost Physics Lecture Notes (2020), doi:10.21468/scipostphyslectnotes.19.

[10] J.-S. Caux, *Correlation functions of integrable models: A description of the ABACUS algorithm*, Journal of Mathematical Physics **50**(9), 095214 (2009), doi:10.1063/1.3216474.

[11] T. Giamarchi, *Quantum Physics in One Dimension*, Oxford University Press, doi:10.1093/acprof:oso/9780198525004.001.0001 (2003).

[12] A. Shashi, L. I. Glazman, J.-S. Caux and A. Imambekov, *Nonuniversal prefactors in the correlation functions of one-dimensional quantum liquids*, Phys. Rev. B **84**, 045408 (2011), doi:10.1103/PhysRevB.84.045408.

[13] A. Shashi, M. Panfil, J.-S. Caux and A. Imambekov, *Exact prefactors in static and dynamic correlation functions of one-dimensional quantum integrable models: Applications to the Calogero-Sutherland, Lieb-Liniger, and XXZ models*, Phys. Rev. B **85**, 155136 (2012), doi:10.1103/PhysRevB.85.155136.

[14] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov and V. Terras, *A form factor approach to the asymptotic behavior of correlation functions in critical models*, Journal of Statistical Mechanics: Theory and Experiment **2011**(12), P12010 (2011), doi:10.1088/1742-5468/2011/12/P12010.

[15] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov and V. Terras, *Form factor approach to dynamical correlation functions in critical models*, Journal of Statistical Mechanics: Theory and Experiment **2012**(09), P09001 (2012), doi:10.1088/1742-5468/2012/09/P09001.

[16] K. K. Kozlowski and J. M. Maillet, *Microscopic approach to a class of 1d quantum critical models*, Journal of Physics A: Mathematical and Theoretical **48**(48), 484004 (2015), doi:10.1088/1751-8113/48/48/484004.

[17] A. Imambekov and L. I. Glazman, *Universal Theory of Nonlinear Luttinger Liquids*, Science **323**(5911), 228 (2009), doi:10.1126/science.1165403.

[18] A. Imambekov, T. L. Schmidt and L. I. Glazman, *One-dimensional quantum liquids: Beyond the Luttinger liquid paradigm*, Rev. Mod. Phys. **84**, 1253 (2012), doi:10.1103/RevModPhys.84.1253.

[19] L. Markhof, M. Pletyukhov and V. Meden, *Investigating the roots of the nonlinear Luttinger liquid phenomenology*, SciPost Phys. **7**, 47 (2019), doi: 10.21468/SciPostPhys.7.4.047.

[20] M. Panfil and J.-S. Caux, *Finite-temperature correlations in the Lieb-Liniger one-dimensional Bose gas*, Phys. Rev. A **89**, 033605 (2014), doi: 10.1103/PhysRevA.89.033605.

[21] J.-S. Caux, *The Quench Action*, Journal of Statistical Mechanics: Theory and Experiment **2016**(6), 064006 (2016), doi:10.1088/1742-5468/2016/06/064006.

[22] T. Price, *Long time thermal asymptotics of nonlinear Luttinger liquid from inverse scattering*, arXiv preprint arXiv:1708.02170 (2017).

[23] C. Karrasch, R. G. Pereira and J. Sirker, *Low temperature dynamics of nonlinear Luttinger liquids*, New Journal of Physics **17**(10), 103003 (2015), doi:10.1088/1367-2630/17/10/103003.

[24] F. Göhmann, *Statistical mechanics of integrable quantum spin systems*, SciPost Phys. Lect. Notes p. 16 (2020), doi:10.21468/SciPostPhysLectNotes.16.

[25] M. Dugave, F. Göhmann and K. K. Kozlowski, *Thermal form factors of theXXZchain and the large-distance asymptotics of its temperature dependent correlation functions*, Journal of Statistical Mechanics: Theory and Experiment **2013**(07), P07010 (2013), doi:10.1088/1742-5468/2013/07/p07010.

[26] M. Dugave, F. Göhmann and K. K. Kozlowski, *Low-temperature large-distance asymptotics of the transversal two-point functions of the XXZ chain*, Journal of Statistical Mechanics: Theory and Experiment **2014**(4), P04012 (2014), doi:10.1088/1742-5468/2014/04/p04012.

[27] F. Göhmann, M. Karbach, A. Klümper, K. K. Kozlowski and J. Suzuki, *Thermal form-factor approach to dynamical correlation functions of inte-*

grable lattice models, Journal of Statistical Mechanics: Theory and Experiment **2017**(11), 113106 (2017), doi:10.1088/1742-5468/aa9678.

[28] A. LeClair and G. Mussardo, *Finite temperature correlation functions in integrable QFT*, Nuclear Physics B **552**(3), 624 (1999), doi:https://doi.org/10.1016/S0550-3213(99)00280-1.

[29] H. Saleur, *A comment on finite temperature correlations in integrable QFT*, Nuclear Physics B **567**(3), 602 (2000), doi:https://doi.org/10.1016/S0550-3213(99)00665-3.

[30] G. Mussardo, *On the finite temperature formalism in integrable quantum field theories*, Journal of Physics A: Mathematical and General **34**(36), 7399 (2001), doi:10.1088/0305-4470/34/36/319.

[31] O. Castro-Alvaredo and A. Fring, *Finite temperature correlation functions from form factors*, Nuclear Physics B **636**(3), 611 (2002), doi:10.1016/s0550-3213(02)00409-1.

[32] B. Doyon, *Finite-temperature form factors in the free Majorana theory*, Journal of Statistical Mechanics: Theory and Experiment **2005**(11), P11006 (2005), doi:10.1088/1742-5468/2005/11/p11006.

[33] B. Altshuler, R. Konik and A. Tsvelik, *Low temperature correlation functions in integrable models: Derivation of the large distance and time asymptotics from the form factor expansion*, Nuclear Physics B **739**(3), 311 (2006), doi:10.1016/j.nuclphysb.2006.01.022.

[34] B. Doyon, *Finite-Temperature Form Factors: a Review*, Symmetry, Integrability and Geometry: Methods and Applications (2007), doi:10.3842/sigma.2007.011.

[35] F. H. L. Essler and R. M. Konik, *Finite-temperature dynamical correlations in massive integrable quantum field theories*, Journal of Statistical Mechanics: Theory and Experiment **2009**(09), P09018 (2009), doi:10.1088/1742-5468/2009/09/p09018.

[36] B. Pozsgay and G. Takács, *Form factor expansion for thermal correlators*, Journal of Statistical Mechanics: Theory and Experiment **2010**(11), P11012 (2010), doi:10.1088/1742-5468/2010/11/p11012.

[37] J. D. Nardis and M. Panfil, *Density form factors of the 1D Bose gas for finite entropy states*, Journal of Statistical Mechanics: Theory and Experiment **2015**(2), P02019 (2015), doi:10.1088/1742-5468/2015/02/p02019.

[38] J. D. Nardis and M. Panfil, *Exact correlations in the Lieb-Liniger model and detailed balance out-of-equilibrium*, SciPost Phys. **1**, 015 (2016), doi: 10.21468/SciPostPhys.1.2.015.

[39] J. D. Nardis and M. Panfil, *Particle-hole pairs and density–density correlations in the Lieb–Liniger model*, Journal of Statistical Mechanics: Theory and Experiment **2018**(3), 033102 (2018), doi:10.1088/1742-5468/aab012.

[40] M. Panfil, *The two particle-hole pairs contribution to the dynamic correlation functions of quantum integrable models*, arXiv preprint arXiv:2008.08872 (2020).

[41] I. Bloch, *Ultracold quantum gases in optical lattices*, Nature Physics **1**(1), 23 (2005), doi:10.1038/nphys138.

[42] S. Hofferberth, I. Lesanovsky, B. Fischer, T. Schumm and J. Schmiedmayer, *Non-equilibrium coherence dynamics in one-dimensional bose gases*, Nature **449**(7160), 324 (2007), doi:10.1038/nature06149.

[43] A. Polkovnikov, K. Sengupta, A. Silva and M. Vengalattore, *Colloquium: Nonequilibrium dynamics of closed interacting quantum systems*, Rev. Mod. Phys. **83**, 863 (2011), doi:10.1103/RevModPhys.83.863.

[44] T. Kinoshita, T. Wenger and D. S. Weiss, *A quantum Newton's cradle*, Nature **440**(7086), 900 (2006), doi:10.1038/nature04693.

[45] J.-S. Caux and F. H. L. Essler, *Time evolution of local observables after quenching to an integrable model*, Physical Review Letters **110**(25) (2013), doi:10.1103/physrevlett.110.257203.

[46] O. A. Castro-Alvaredo, B. Doyon and T. Yoshimura, *Emergent hydrodynamics in integrable quantum systems out of equilibrium*, Physical Review X **6**(4) (2016), doi:10.1103/physrevx.6.041065.

[47] B. Bertini, M. Collura, J. De Nardis and M. Fagotti, *Transport in out-of-equilibrium XXZ chains: Exact profiles of charges and currents*, Phys. Rev. Lett. **117**, 207201 (2016), doi:10.1103/PhysRevLett.117.207201.

[48] J. D. Nardis, B. Doyon, M. Medenjak and M. Panfil, *Correlation functions and transport coefficients in generalised hydrodynamics*, Journal of Statistical Mechanics: Theory and Experiment **2022**(1), 014002 (2022), doi:10.1088/1742-5468/ac3658.

[49] B. J. van Wees, H. van Houten, C. W. J. Beenakker, J. G. Williamson, L. P. Kouwenhoven, D. van der Marel and C. T. Foxon, *Quantized conductance of point contacts in a two-dimensional electron gas*, Physical Review Letters **60**(9), 848–850 (1988), doi:10.1103/physrevlett.60.848.

[50] R. Landauer, *Spatial variation of currents and fields due to localized scatterers in metallic conduction*, IBM Journal of Research and Development **1**(3), 223 (1957), doi:10.1147/rd.13.0223.

[51] R. Landauer, *Electrical resistance of disordered one-dimensional lattices*, The Philosophical Magazine: A Journal of Theoretical Experimental and Applied Physics **21**(172), 863 (1970), doi:10.1080/14786437008238472, `https://doi.org/10.1080/14786437008238472`.

[52] M. Büttiker, *Four-terminal phase-coherent conductance*, Phys. Rev. Lett. **57**, 1761 (1986), doi:10.1103/PhysRevLett.57.1761.

[53] R. Landauer, *Conductance from transmission: common sense points*, Physica Scripta **T42**, 110 (1992), doi:10.1088/0031-8949/1992/t42/020.

[54] Y. Imry and R. Landauer, *Conductance viewed as transmission*, Reviews of Modern Physics **71**(2), S306 (1999), doi:10.1103/revmodphys.71.s306.

[55] P. Calabrese and J. Cardy, *Entanglement and correlation functions following a local quench: a conformal field theory approach*, Journal of Statistical Mechanics: Theory and Experiment **2007**(10), P10004 (2007), doi:10.1088/1742-5468/2007/10/p10004.

[56] S. Sotiriadis and J. Cardy, *Inhomogeneous quantum quenches*, Journal of Statistical Mechanics: Theory and Experiment **2008**(11), P11003 (2008), doi:10.1088/1742-5468/2008/11/p11003.

[57] A. Polkovnikov, K. Sengupta, A. Silva and M. Vengalattore, *Colloquium: Nonequilibrium dynamics of closed interacting quantum systems*, Rev. Mod. Phys. **83**, 863 (2011).

[58] P. Calabrese, F. H. L. Essler and G. Mussardo, *Introduction to 'quantum integrability in out of equilibrium systems'*, Journal of Statistical Mechanics: Theory and Experiment **2016**(6), 064001 (2016), doi:10.1088/1742-5468/2016/06/064001.

[59] J. Eisert, M. Friesdorf and C. Gogolin, *Quantum many-body systems out of equilibrium*, Nature Physics **11**(2), 124 (2015), doi:10.1038/nphys3215.

[60] C. Caroli, R. Combescot, P. Nozieres and D. Saint-James, *Direct calculation of the tunneling current*, Journal of Physics C: Solid State Physics **4**(8), 916 (1971), doi:10.1088/0022-3719/4/8/018.

[61] G. Stefanucci and C.-O. Almbladh, *Time-dependent partition-free approach in resonant tunneling systems*, Phys. Rev. B **69**, 195318 (2004), doi:10.1103/PhysRevB.69.195318.

[62] B. Bertini, M. Collura, J. D. Nardis and M. Fagotti, *Transport in out-of-EquilibriumXXZChains: Exact profiles of charges and currents*, Physical Review Letters **117**(20) (2016), doi:10.1103/physrevlett.117.207201.

[63] A. Bastianello, B. Bertini, B. Doyon and R. Vasseur, *Introduction to the special issue on emergent hydrodynamics in integrable many-body systems*, Journal of Statistical Mechanics: Theory and Experiment **2022**(1), 014001 (2022), doi:10.1088/1742-5468/ac3e6a.

[64] F. Bornemann, *On the numerical evaluation of fredholm determinants*, Mathematics of Computation **79**(270), 871 (2009), doi:10.1090/s0025-5718-09-02280-7.

[65] R. E. Hartwig and M. E. Fisher, *Asymptotic behavior of Toeplitz matrices and determinants*, Archive for Rational Mechanics and Analysis **32**(3), 190 (1969), doi:10.1007/bf00247509.

[66] E. Barouch and B. M. McCoy, *Statistical Mechanics of the XY Model. II. Spin-Correlation Functions*, Phys. Rev. A **3**, 786 (1971), doi:10.1103/PhysRevA.3.786.

[67] O. I. Pâţu, *Correlation functions and momentum distribution of one-dimensional hard-core anyons in optical lattices*, Journal of Statistical Mechanics: Theory and Experiment **2015**(1), P01004 (2015), doi:10.1088/1742-5468/2015/01/p01004.

[68] X. Jie, *The large time asymptotics of the temperature correlation functions of the xxo heisenberg ferromagnet. the riemann-hilbert approach.*, Ph.D. thesis, Indiana University - Purdue University Ph.D. Thesis (1998).

[69] S. Sachdev and A. P. Young, *Low Temperature Relaxational Dynamics of the Ising Chain in a Transverse Field*, Phys. Rev. Lett. **78**, 2220 (1997), doi:10.1103/PhysRevLett.78.2220.

[70] E. Granet, M. Fagotti and F. H. L. Essler, *Finite temperature and quench dynamics in the Transverse Field Ising Model from form factor expansions*, SciPost Phys. **9**, 33 (2020), doi:10.21468/SciPostPhys.9.3.033.

[71] Etienne Granet and Fabian H. L. Essler, *A systematic* $1/c$*-expansion of form factor sums for dynamical correlations in the Lieb-Liniger model*, arXiv preprint arXiv:2007.15396 (2020).

[72] F. Colomo, A. Izergin, V. Korepin and V. Tognetti, *Correlators in the Heisenberg XXO chain as Fredholm determinants*, Physics Letters A **169**(4), 243 (1992), doi:10.1016/0375-9601(92)90452-r.

[73] F. Colomo, A. G. Izergin, V. E. Korepin and V. Tognetti, *Temperature correlation functions in the XX0 Heisenberg chain. I*, Theoretical and Mathematical Physics **94**(1), 11 (1993), doi:10.1007/bf01016992.

[74] F. Colomo, A. G. Izergin and V. Tognetti, *Correlation functions in the XXO Heisenberg chain and their relations with spectral shapes*, Journal of Physics A: Mathematical and General **30**(2), 361 (1997), doi:10.1088/0305-4470/30/2/004.

[75] A. G. Izergin, V. S. Kapitonov and N. A. Kitanin, *Equal-time temperature correlators of the one-dimensional Heisenberg XY chain*, Journal of Mathematical Sciences **100**(2), 2120 (2000), doi:10.1007/bf02675733.

[76] A. I. Bugrii and O. O. Lisovyy, *Correlation Function of the Two-Dimensional Ising Model on a Finite Lattice: II*, Theoretical and Mathematical Physics **140**(1), 987 (2004), doi:10.1023/b:tamp.0000033035.90327.1f.

[77] G. von Gehlen, N. Iorgov, S. Pakuliak, V. Shadura and Y. Tykhyy, *Form-factors in the Baxter–Bazhanov–Stroganov model II: Ising model on the finite lattice*, Journal of Physics A: Mathematical and Theoretical **41**(9), 095003 (2008), doi:10.1088/1751-8113/41/9/095003.

[78] N. Iorgov and O. Lisovyy, *Ising correlations and elliptic determinants*, Journal of Statistical Physics **143**(1), 33 (2011), doi:10.1007/s10955-011-0154-6.

[79] N. Iorgov and O. Lisovyy, *Finite-lattice form factors in free-fermion models*, Journal of Statistical Mechanics: Theory and Experiment **2011**(04), P04011 (2011), doi:10.1088/1742-5468/2011/04/p04011.

[80] N. Iorgov, *Form factors of the finite quantum XY-chain*, Journal of Physics A: Mathematical and Theoretical **44**(33), 335005 (2011), doi:10.1088/1751-8113/44/33/335005.

[81] N. A. Slavnov, *Integral operators with the generalized sine kernel on the real axis*, Theoretical and Mathematical Physics **165**(1), 1262 (2010), doi:10.1007/s11232-010-0108-1.

[82] V. Korepin and N. Slavnov, *Time and temperature dependent correlation functions of 1d models of quantum statistical mechanics*, Physics Letters A **236**(3), 201 (1997), doi:10.1016/s0375-9601(97)00800-1.

[83] P. Deift, *Integrable operators*, American Mathematical Society Translations **189**(2), 69 (1999).

[84] E. Lieb, T. Schultz and D. Mattis, *Two soluble models of an antiferromagnetic chain*, Annals of Physics **16**(3), 407 (1961), doi:10.1016/0003-4916(61)90115-4.

[85] O. Gamayun, A. G. Pronko and M. B. Zvonarev, *Time and temperature-dependent correlation function of an impurity in one-dimensional Fermi and Tonks–Girardeau gases as a Fredholm determinant*, New Journal of Physics **18**(4), 045005 (2016), doi:10.1088/1367-2630/18/4/045005.

[86] P. W. Anderson, *Infrared catastrophe in fermi gases with local scattering potentials*, Phys. Rev. Lett. **18**, 1049 (1967), doi:10.1103/PhysRevLett.18.1049.

[87] A. Lenard, *Momentum Distribution in the Ground State of the One-Dimensional System of Impenetrable Bosons*, Journal of Mathematical Physics **5**(7), 930 (1964), doi:10.1063/1.1704196.

[88] A. Lenard, *One-Dimensional Impenetrable Bosons in Thermal Equilibrium*, Journal of Mathematical Physics **7**(7), 1268 (1966), doi:10.1063/1.1705029.

[89] P. Forrester, N. Frankel, T. Garoni and N. Witte, *Painlevé Transcendent Evaluations of Finite System Density Matrices for 1d Impenetrable Bosons*, Communications in Mathematical Physics **238**(1), 257 (2003), doi:10.1007/s00220-003-0851-3.

[90] S. Katsura, *Statistical Mechanics of the Anisotropic Linear Heisenberg Model*, Phys. Rev. **127**, 1508 (1962), doi:10.1103/PhysRev.127.1508.

[91] S. Sachdev, *Universal, finite-temperature, crossover functions of the quantum transition in the ising chain in a transverse field*, Nuclear Physics B **464**(3), 576 (1996), doi:10.1016/0550-3213(95)00657-5.

[92] G. Szegő, *Ein Grenzwertsatz uber die Toeplitzschen Determinanten einer reellen positiven Funktion*, Mathematische Annalen **76**(4), 490 (1915), doi: 10.1007/bf01458220.

[93] U. Grenander and G. Szegő, *Toeplitz Forms and Their Applications*, AMS Chelsea Publishing Series. American Mathematical Society, ISBN 9780821828441 (2001).

[94] P. A. Deift, A. R. Its and X. Zhou, *A Riemann-Hilbert Approach to Asymptotic Problems Arising in the Theory of Random Matrix Models, and also in the Theory of Integrable Statistical Mechanics*, The Annals of Mathematics **146**(1), 149 (1997), doi:10.2307/2951834.

[95] P. Calabrese, F. H. L. Essler and M. Fagotti, *Quantum Quench in the Transverse-Field Ising Chain*, Phys. Rev. Lett. **106**, 227203 (2011), doi: 10.1103/PhysRevLett.106.227203.

[96] P. Calabrese, F. H. L. Essler and M. Fagotti, *Quantum quench in the transverse field Ising chain: I. Time evolution of order parameter correlators*, Journal of Statistical Mechanics: Theory and Experiment **2012**(07), P07016 (2012), doi:10.1088/1742-5468/2012/07/p07016.

[97] P. Calabrese, F. H. L. Essler and M. Fagotti, *Quantum quenches in the transverse field Ising chain: II. Stationary state properties*, Journal of Statistical Mechanics: Theory and Experiment **2012**(07), P07022 (2012), doi: 10.1088/1742-5468/2012/07/p07022.

[98] O. I. Pâţu, V. E. Korepin and D. V. Averin, *Correlation functions of one-dimensional Lieb–Liniger anyons*, Journal of Physics A: Mathematical and Theoretical **40**(50), 14963 (2007), doi:10.1088/1751-8113/40/50/004.

[99] O. I. Pâţu, V. E. Korepin and D. V. Averin, *One-dimensional impenetrable anyons in thermal equilibrium: I. Anyonic generalization of Lenard's formula*, Journal of Physics A: Mathematical and Theoretical **41**(14), 145006 (2008), doi:10.1088/1751-8113/41/14/145006.

[100] O. I. Pâţu, V. E. Korepin and D. V. Averin, *Large-distance asymptotic behavior of the correlation functions of 1D impenetrable anyons at finite temperatures*, EPL (Europhysics Letters) **86**(4), 40001 (2009), doi:10.1209/0295-5075/86/40001.

[101] O. I. Pâţu, V. E. Korepin and D. V. Averin, *One-dimensional impenetrable anyons in thermal equilibrium: III. Large distance asymptotics of the space correlations*, Journal of Physics A: Mathematical and Theoretical **42**(27), 275207 (2009), doi:10.1088/1751-8113/42/27/275207.

[102] O. I. Pâţu, *Correlation functions and momentum distribution of one-dimensional hard-core anyons in optical lattices*, Journal of Statistical Mechanics: Theory and Experiment **2015**(1), P01004 (2015), doi:10.1088/1742-5468/2015/01/p01004.

[103] O. Gamayun, A. G. Pronko and M. B. Zvonarev, *Impurity Green's function of a one-dimensional Fermi gas*, Nuclear Physics B **892**, 83 (2015), doi:10.1016/j.nuclphysb.2015.01.004.

[104] O. Gamayun, O. Lychkovskiy, E. Burovski, M. Malcomson, V. V. Cheianov and M. B. Zvonarev, *Impact of the Injection Protocol on an Impurity's Stationary State*, Phys. Rev. Lett. **120**, 220605 (2018), doi:10.1103/PhysRevLett.120.220605.

[105] F. Göhmann, K. K. Kozlowski and J. Suzuki, *High-temperature analysis of the transverse dynamical two-point correlation function of the XX quantum-spin chain*, Journal of Mathematical Physics **61**(1), 013301 (2020), doi:10.1063/1.5111039.

[106] F. Göhmann, K. K. Kozlowski, J. Sirker and J. Suzuki, *Equilibrium dynamics of the XX chain*, Physical Review B **100**(15) (2019), doi:10.1103/physrevb.100.155428.

[107] F. Göhmann, K. K. Kozlowski and J. Suzuki, *Long-time large-distance asymptotics of the transverse correlation functions of the XX chain in the spacelike regime*, Letters in Mathematical Physics **110**(7), 1783 (2020), doi:10.1007/s11005-020-01276-y.

[108] F. Gohmann and A. Seel, *The XX and Ising Limits in Integral Formulas for Finite-Temperature Correlation Functions of the XXZ Chain*, Theoretical and Mathematical Physics **146**(1), 119 (2006), doi:10.1007/s11232-006-0012-x.

[109] N. Kitanine, K. K. Kozlowski, J. M. Maillet, N. A. Slavnov and V. Terras, *Riemann–Hilbert Approach to a Generalised Sine Kernel and Applications*, Communications in Mathematical Physics **291**(3), 691 (2009), doi:10.1007/s00220-009-0878-1.

[110] A. R. Its, A. G. Izergin, V. E. Korepin and N. A. Slavnov, *Temperature Correlations of Quantum Spins*, Phys. Rev. Lett. **70**, 2357 (1993), doi:10.1103/PhysRevLett.70.2357.

[111] O. I. Pâţu, V. E. Korepin and D. V. Averin, *Correlation functions of one-dimensional Lieb–Liniger anyons*, Journal of Physics A: Mathematical and Theoretical **40**(50), 14963 (2007), doi:10.1088/1751-8113/40/50/004.

[112] O. I. Pâţu, V. E. Korepin and D. V. Averin, *One-dimensional impenetrable anyons in thermal equilibrium: I. anyonic generalization of lenard's formula*, Journal of Physics A: Mathematical and Theoretical **41**(14), 145006 (2008), doi:10.1088/1751-8113/41/14/145006.

[113] O. I. Pâţu, V. E. Korepin and D. V. Averin, *One-dimensional impenetrable anyons in thermal equilibrium: II. determinant representation for the dynamic correlation functions*, Journal of Physics A: Mathematical and Theoretical **41**(25), 255205 (2008), doi:10.1088/1751-8113/41/25/255205.

[114] O. I. Pâţu, V. E. Korepin and D. V. Averin, *One-dimensional impenetrable anyons in thermal equilibrium: III. large distance asymptotics of the space correlations*, Journal of Physics A: Mathematical and Theoretical **42**(27), 275207 (2009), doi:10.1088/1751-8113/42/27/275207.

[115] N. T. Zinner, *Strongly interacting mesoscopic systems of anyons in one dimension*, Phys. Rev. A **92**, 063634 (2015), doi:10.1103/PhysRevA.92.063634.

[116] T. Keilmann, S. Lanzmich, I. McCulloch and M. Roncaglia, *Statistically induced phase transitions and anyons in 1d optical lattices*, Nature Communications **2**, 361 (2011), doi:10.1038/ncomms1353.

[117] S. Greschner, L. Cardarelli and L. Santos, *Probing the exchange statistics of one-dimensional anyon models*, Phys. Rev. A **97**, 053605 (2018), doi:10.1103/PhysRevA.97.053605.

[118] L. Cardarelli, S. Greschner and L. Santos, *Engineering interactions and anyon statistics by multicolor lattice-depth modulations*, Phys. Rev. A **94**, 023615 (2016), doi:10.1103/PhysRevA.94.023615.

[119] C. Sträter, S. C. L. Srivastava and A. Eckardt, *Floquet realization and signatures of one-dimensional anyons in an optical lattice*, Phys. Rev. Lett. **117**, 205303 (2016), doi:10.1103/PhysRevLett.117.205303.

[120] S. Greschner and L. Santos, *Anyon Hubbard Model in One-Dimensional Optical Lattices*, Phys. Rev. Lett. **115**, 053002 (2015), doi:10.1103/PhysRevLett.115.053002.

[121] M. Aguado, G. K. Brennen, F. Verstraete and J. I. Cirac, *Creation, manipulation, and detection of abelian and non-abelian anyons in optical lattices*, Phys. Rev. Lett. **101**, 260501 (2008), doi:10.1103/PhysRevLett.101.260501.

[122] L.-M. Duan, E. Demler and M. D. Lukin, *Controlling spin exchange interactions of ultracold atoms in optical lattices*, Phys. Rev. Lett. **91**, 090402 (2003), doi:10.1103/PhysRevLett.91.090402.

[123] A. Micheli, G. K. Brennen and P. Zoller, *A toolbox for lattice-spin models with polar molecules*, Nature Physics **2**, 341–347 (2006), doi:10.1038/nphys287.

[124] L. Jiang, G. K. Brennen, A. V. Gorshkov, K. Hammerer, M. Hafezi, E. Demler, M. D. Lukin and P. Zoller, *Anyonic interferometry and protected memories in atomic spin lattices*, Nature Physics **4**, 482–488 (2008), doi:10.1038/nphys943.

[125] A. Berkovich and J. Lowenstein, *Correlation function of the one-dimensional fermi gas in the infinite-coupling limit (repulsive case)*, Nuclear Physics B **285**, 70 (1987), doi:10.1016/0550-3213(87)90329-4.

[126] A. Berkovich, *Temperature and magnetic field-dependent correlators of the exactly integrable (1+1)-dimensional gas of impenetrable fermions*, Journal of Physics A: Mathematical and General **24**(7), 1543 (1991), doi:10.1088/0305-4470/24/7/025.

[127] F. Göhmann, A. Its and V. Korepin, *Correlations in the impenetrable electron gas*, Physics Letters A **249**(1-2), 117 (1998), doi:10.1016/s0375-9601(98)00711-7.

[128] A. Izergin and A. Pronko, *Temperature correlators in the two-component one-dimensional gas*, Nuclear Physics B **520**(3), 594 (1998), doi:https://doi.org/10.1016/S0550-3213(98)00182-5.

[129] F. Göhmann, A. G. Izergin, V. E. Korepin and A. G. Pronko, *Time and temperature dependent correlation functions of the one-dimensional impenetrable electron gas*, International Journal of Modern Physics B **12**(23), 2409 (1998), doi:10.1142/s0217979298001411.

[130] V. V. Cheianov and M. B. Zvonarev, *Nonunitary spin-charge separation in a one-dimensional fermion gas*, Phys. Rev. Lett. **92**, 176401 (2004), doi:10.1103/PhysRevLett.92.176401.

[131] G. A. Fiete and L. Balents, *Green's function for magnetically incoherent interacting electrons in one dimension*, Phys. Rev. Lett. **93**, 226401 (2004), doi:10.1103/PhysRevLett.93.226401.

[132] V. V. Cheianov and M. B. Zvonarev, *Zero temperature correlation functions for the impenetrable fermion gas*, Journal of Physics A: Mathematical and General **37**(6), 2261 (2004), doi:10.1088/0305-4470/37/6/021.

[133] O. I. Pâţu, *Correlation functions of one-dimensional strongly interacting two-component gases*, Phys. Rev. A **100**, 063635 (2019), doi:10.1103/PhysRevA.100.063635.

[134] M. B. Zvonarev, V. V. Cheianov and T. Giamarchi, *The time-dependent correlation function of the jordan–wigner operator as a fredholm determinant*, Journal of Statistical Mechanics: Theory and Experiment **2009**(07), P07035 (2009), doi:10.1088/1742-5468/2009/07/p07035.

[135] O. Gamayun, A. G. Pronko and M. B. Zvonarev, *Impurity Green's function of a one-dimensional Fermi gas*, Nuclear Physics B **892**, 83 (2015), doi: http://dx.doi.org/10.1016/j.nuclphysb.2015.01.004.

[136] O. Gamayun, O. Lychkovskiy, E. Burovski, M. Malcomson, V. V. Cheianov and M. B. Zvonarev, *Impact of the injection protocol on an impurity's stationary state*, Phys. Rev. Lett. **120**, 220605 (2018), doi:10.1103/PhysRevLett.120.220605.

[137] O. Gamayun, O. Lychkovskiy and M. B. Zvonarev, *Zero temperature momentum distribution of an impurity in a polaron state of one-dimensional Fermi and Tonks–Girardeau gases*, SciPost Phys. **8**, 53 (2020), doi: 10.21468/SciPostPhys.8.4.053.

[138] A. Imambekov and L. I. Glazman, *Universal Theory of Nonlinear Luttinger Liquids*, Science **323**(5911), 228 (2009), doi:10.1126/science.1165403.

[139] D. Chernowitz and O. Gamayun, *On the Dynamics of Free-Fermionic Tau-Functions at Finite Temperature*, SciPost Phys. Core **5**, 6 (2022), doi: 10.21468/SciPostPhysCore.5.1.006.

[140] A. R. Its, A. G. Izergin, V. E. Korepin and N. A. Slavnov, *Temperature correlations of quantum spins*, Phys. Rev. Lett. **70**, 1704 (1993), doi: 10.1103/PhysRevLett.70.1704.

[141] V. E. Zakharov and S. V. Manakov, *Asymptotic Behavior of Nonlinear Wave Systems Integrated by the Method of the Inverse Scattering Problem*, Zh. Eksp. Teor. Fiz. **71**, 203 (1976).

[142] R. F. Bikbaev and A. Its, *Asymptotics at $t \to \infty$ of the solution of the cauchy problem for the landau-lifshitz equation*, Theoretical and Mathematical Physics **76**, 665 (1988).

[143] P. A. Deift, A. R. Its and X. Zhou, *Long-Time Asymptotics for Integrable Nonlinear Wave Equations*, p. 181–204, Springer Berlin Heidelberg, ISBN 9783642580451, doi:10.1007/978-3-642-58045-1_10 (1993).

[144] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer Berlin Heidelberg, doi:10.1007/978-3-540-69969-9 (1987).

[145] S. Novikov, S. Manakov, L. Pitaevskii and V. Zakharov, *Theory of solitons: the inverse scattering method*, Publisher Springer Verlag New York LLC (1984).

[146] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge Monographs on Mathematical Physics. Cambridge University Press, doi:10.1017/CBO9780511628832 (1993).

[147] O. I. Pâţu, V. E. Korepin and D. V. Averin, *Large-distance asymptotic behavior of the correlation functions of 1d impenetrable anyons at finite temperatures*, EPL (Europhysics Letters) **86**(4), 40001 (2009), doi:10.1209/0295-5075/86/40001.

[148] O. I. Pâţu, V. E. Korepin and D. V. Averin, *One-dimensional impenetrable anyons in thermal equilibrium: IV. large time and distance asymptotic behavior of the correlation functions*, Journal of Physics A: Mathematical and Theoretical **43**(11), 115204 (2010), doi:10.1088/1751-8113/43/11/115204.

[149] R. Santachiara and P. Calabrese, *One-particle density matrix and momentum distribution function of one-dimensional anyon gases*, Journal of Statistical Mechanics: Theory and Experiment **2008**(06), P06005 (2008), doi:10.1088/1742-5468/2008/06/p06005.

[150] S. Scopa, L. Piroli and P. Calabrese, *One-particle density matrix of a trapped lieb–liniger anyonic gas*, Journal of Statistical Mechanics: Theory and Experiment **2020**(9), 093103 (2020), doi:10.1088/1742-5468/abaed1.

[151] K. K. Kozlowski, J. M. Maillet and N. A. Slavnov, *Correlation functions for one-dimensional bosons at low temperature*, Journal of Statis-

tical Mechanics: Theory and Experiment **2011**(03), P03019 (2011), doi: 10.1088/1742-5468/2011/03/p03019.

[152] L. S. Levitov and G. B. Lesovik, *Shot noise measurements in a wide-channel transistor near pinch-off*, JETP Letters **58**, 230 (1993).

[153] L. S. Levitov, H. Lee and G. B. Lesovik, *Electron counting statistics and coherent states of electric current*, Journal of Mathematical Physics **37**(10), 4845 (1996), doi:10.1063/1.531672.

[154] K. Schönhammer, *Full counting statistics for noninteracting fermions: Exact results and the levitov-lesovik formula*, Physical Review B **75**(20) (2007), doi:10.1103/physrevb.75.205329.

[155] E. Khosravi, S. Kurth, G. Stefanucci and E. K. U. Gross, *The role of bound states in time-dependent quantum transport*, Applied Physics A **93**(2), 355 (2008), doi:10.1007/s00339-008-4864-9.

[156] P.-Y. Yang, C.-Y. Lin and W.-M. Zhang, *Master equation approach to transient quantum transport in nanostructures incorporating initial correlations*, Phys. Rev. B **92**, 165403 (2015), doi:10.1103/PhysRevB.92.165403.

[157] M. Büttiker, *Scattering theory of current and intensity noise correlations in conductors and wave guides*, Phys. Rev. B **46**, 12485 (1992), doi:10.1103/PhysRevB.46.12485.

[158] R. G. Newton, *Scattering Theory of Waves and Particles*, Springer Berlin Heidelberg, doi:10.1007/978-3-642-88128-2 (1982).

[159] S. P. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of Solitons, the Inverse Scattering Method*, Consultants Bureau, New York (1984).

[160] O. Gamayun, O. Lychkovskiy and J.-S. Caux, *Fredholm determinants, full counting statistics and loschmidt echo for domain wall profiles in one-dimensional free fermionic chains*, SciPost Physics **8**(3) (2020), doi:10.21468/scipostphys.8.3.036.

[161] K. K. Kozlowski, *Riemann-hilbert approach to the time-dependent generalized sine kernel*, Advances in Theoretical and Mathematical Physics **15**(6), 1655 (2011), doi:10.4310/atmp.2011.v15.n6.a3.

[162] M. Esposito, U. Harbola and S. Mukamel, *Nonequilibrium fluctuations, fluctuation theorems, and counting statistics in quantum systems*, Rev. Mod. Phys. **81**, 1665 (2009), doi:10.1103/RevModPhys.81.1665.

[163] B. Doyon and J. Myers, *Fluctuations in ballistic transport from euler hydrodynamics*, Annales Henri Poincaré **21**(1), 255 (2019), doi: 10.1007/s00023-019-00860-w.

[164] D. B. Gutman, Y. Gefen and A. D. Mirlin, *Bosonization of one-dimensional fermions out of equilibrium*, Phys. Rev. B **81**, 085436 (2010), doi:10.1103/PhysRevB.81.085436.

[165] D. B. Gutman, Y. Gefen and A. D. Mirlin, *Non-equilibrium 1d many-body problems and asymptotic properties of toeplitz determinants*, Journal of Physics A: Mathematical and Theoretical **44**(16), 165003 (2011), doi: 10.1088/1751-8113/44/16/165003.

[166] I. Klich and L. Levitov, *Quantum noise as an entanglement meter*, Physical Review Letters **102**(10) (2009), doi:10.1103/physrevlett.102.100502.

[167] I. Klich and L. Levitov, *Many-body entanglement: a new application of the full counting statistics*, In *AIP Conference Proceedings*. AIP, doi:10.1063/1.3149497 (2009).

[168] H. F. Song, C. Flindt, S. Rachel, I. Klich and K. L. Hur, *Entanglement entropy from charge statistics: Exact relations for noninteracting many-body systems*, Physical Review B **83**(16) (2011), doi:10.1103/physrevb.83.161408.

[169] H. F. Song, S. Rachel, C. Flindt, I. Klich, N. Laflorencie and K. L. Hur, *Bipartite fluctuations as a probe of many-body entanglement*, Physical Review B **85**(3) (2012), doi:10.1103/physrevb.85.035409.

[170] P. Calabrese and J. Cardy, *Evolution of entanglement entropy in one-dimensional systems*, Journal of Statistical Mechanics: Theory and Experiment **2005**(04), P04010 (2005), doi:10.1088/1742-5468/2005/04/p04010.

[171] V. Eisler and I. Peschel, *On entanglement evolution across defects in critical chains*, EPL (Europhysics Letters) **99**(2), 20001 (2012), doi: 10.1209/0295-5075/99/20001.

[172] K. Bidzhiev and G. Misguich, *Out-of-equilibrium dynamics in a quantum impurity model: Numerics for particle transport and entanglement entropy*, Physical Review B **96**(19) (2017), doi:10.1103/physrevb.96.195117.

[173] V. Alba and P. Calabrese, *Entanglement and thermodynamics after a quantum quench in integrable systems*, Proceedings of the National Academy of Sciences **114**(30), 7947 (2017), doi:10.1073/pnas.1703516114, `https://www.pnas.org/content/114/30/7947.full.pdf`.

[174] P. Calabrese and J. Cardy, *Entanglement entropy and conformal field theory*, Journal of Physics A: Mathematical and Theoretical **42**(50), 504005 (2009), doi:10.1088/1751-8113/42/50/504005.

[175] I. Peschel and V. Eisler, *Reduced density matrices and entanglement entropy in free lattice models*, Journal of Physics A: Mathematical and Theoretical **42**(50), 504003 (2009), doi:10.1088/1751-8113/42/50/504003.

[176] V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions*, Cambridge University Press, Cambridge (1993).

[177] O. Gamayun, A. Slobodeniuk, J.-S. Caux and O. Lychkovskiy, *Nonequilibrium phase transition in transport through a driven quantum point contact*, Phys. Rev. B **103**, L041405 (2021), doi:10.1103/PhysRevB.103.L041405.

[178] Y. Imry, *Introduction to Mesoscopic Physics*, Mesoscopic Physics and Nanotechnology. Oxford University Press, New York, NY (1996).

[179] B. J. van Wees, H. van Houten, C. W. J. Beenakker, J. G. Williamson, L. P. Kouwenhoven, D. van der Marel and C. T. Foxon, *Quantized conductance*

of point contacts in a two-dimensional electron gas, Phys. Rev. Lett. **60**, 848 (1988), doi:10.1103/PhysRevLett.60.848.

[180] S. Kohler, J. Lehmann and P. Hanggi, *Driven quantum transport on the nanoscale*, Physics Reports **406**(6), 379 (2005), doi:10.1016/j.physrep.2004.11.002.

[181] M. Lebrat, P. Grišins, D. Husmann, S. Häusler, L. Corman, T. Giamarchi, J.-P. Brantut and T. Esslinger, *Band and correlated insulators of cold fermions in a mesoscopic lattice*, Phys. Rev. X **8**, 011053 (2018), doi:10.1103/PhysRevX.8.011053.

[182] Y. Meir and N. S. Wingreen, *Landauer formula for the current through an interacting electron region*, Phys. Rev. Lett. **68**, 2512 (1992), doi:10.1103/PhysRevLett.68.2512.

Appendix A

Summation of form factors and determinant formula

In this appendix, we derive formula (1.12) presenting tau function in the thermodynamic limit as a difference of two Fredholm determinants.

We consider solutions in the large L limit and choose $\mathbf{k}$ to fill a Fermi Sea, namely

$$k_i = \frac{2\pi}{L}\left(-\frac{N}{2} + i - 1 - \nu_i\right), \qquad i = 1, \ldots, N+1, \tag{A.1}$$

where $\nu_i \equiv \nu(k_i)$. For simplicity, we choose N to be even.

First, we identically rewrite the overlap as (note, $\det D = \det \tilde{D}$)

$$|\langle \mathbf{k}|\mathbf{q}\rangle|^2 = -4L \prod_{i=1}^{N+1} \Omega_i \left(\prod_{i=1}^{N+1} \frac{e^{g(k_i)/2}\sin \pi\nu_i}{L}\right)^2 \prod_{i=1}^{N} e^{-g(q_i)} \det D \det \tilde{D} \tag{A.2}$$

$$D = \begin{pmatrix} \cot\frac{k_1-q_1}{2} - i & \ldots & \cot\frac{k_{N+1}-q_1}{2} - i \\ \vdots & \ddots & \vdots \\ \cot\frac{k_1-q_N}{2} - i & \ldots & \cot\frac{k_{N+1}-q_N}{2} - i \\ 1 & \ldots & 1 \end{pmatrix}, \tag{A.3}$$

$$\tilde{D} = \begin{pmatrix} \cot\frac{k_1-q_1}{2} + i & \ldots & \cot\frac{k_{N+1}-q_1}{2} + i \\ \vdots & \ddots & \vdots \\ \cot\frac{k_1-q_N}{2} + i & \ldots & \cot\frac{k_{N+1}-q_N}{2} + i \\ 1 & \ldots & 1 \end{pmatrix}, \tag{A.4}$$

$$\Omega_i = \frac{1}{1 + \frac{2\pi\nu'(k_i)}{L}}. \tag{A.5}$$

Then using standard linear algebra manipulations we rewrite the static tau function as

$$\tau(x) = \det(\mathcal{A} + \delta\mathcal{A}) - \det \mathcal{A} \tag{A.6}$$

with

$$\delta\mathcal{A}_{ij} = -\frac{4\Omega_i}{L}\sin^2(\pi\nu_i)e^{g(k_i)}e^{-i(k_i+k_j)x/2}, \tag{A.7}$$

$$\mathcal{A}_{ij} = \Omega_i\frac{\sin^2(\pi\nu_i)}{L^2}e^{g(k_i)}e^{-i(k_i+k_j)x/2} \\ \times\sum_q e^{iqx-g(q)}\left(\cot\frac{q-k_i}{2}-i\right)\left(\cot\frac{q-k_j}{2}+i\right), \tag{A.8}$$

where summation over q is happening over the whole Brillouin zone

$$q\in\left\{\frac{2\pi}{L}\left(-\frac{L-1}{2}+j-1\right), \qquad j=1,\ldots,L\right\}. \tag{A.9}$$

For $i\neq j$ we present

$$\mathcal{A}_{ij} = \Omega_i\frac{\sin^2\pi\nu_i}{2L}e^{g(k_i)}e^{-i(k_i+k_j)x/2}e^{i(k_i-k_j)/2}\frac{c(k_i)-c(k_j)}{\sin\frac{k_i-k_j}{2}} \tag{A.10}$$

with

$$c(k_i) = \frac{2}{L}\sum_q e^{iqx-g(q)}\cot\frac{q-k_i}{2}. \tag{A.11}$$

This sum can be rewritten as a contour integral and evaluated at large L, namely, choosing contour γ running around q_i and avoiding any other singularities of the integrand we obtain

$$c(k_i) = \oint_\gamma \frac{dq}{\pi}\frac{e^{-g(q)+iqx}}{e^{iqL}-1}\cot\frac{q-k_i}{2}. \tag{A.12}$$

Further, we deform the contour into the rectangle that encapsulates interval $[-\pi,\pi]$. The vertical parts of this rectangle cancel and we are left with two lines above and below the real axis along with the contribution from the pole at $q=k_i$

$$c(k_i) = \left(\int_{-\pi-i0}^{\pi-i0} - \int_{-\pi+i0}^{\pi+i0}\right)\frac{dq}{\pi}\frac{e^{-g(q)+iqx}}{e^{iqL}-1}\cot\frac{q-k_i}{2} - \frac{4ie^{-g(k_i)+ik_ix}}{e^{ik_iL}-1}. \tag{A.13}$$

Here we assume that the imaginary shift $i0$ is chosen to be larger then $\operatorname{Im}k_i = O(1/L)$. In this form we immediately see that, in the limit $L\to\infty$, the values of $c(k)$ at points k_i are equal to the values of the $E(k_i)$ for the analytic function $E(k)$ given by

$$E(k) = \int\limits_{-\pi+i0}^{\pi+i0} \frac{dq}{\pi} e^{-g(q)+iqx} \cot\frac{q-k}{2} - \frac{4ie^{-g(k)+ikx}}{e^{-2\pi i\nu(k)}-1}. \tag{A.14}$$

Using $E(k)$ we can obtain values also for some vicinity of k_i, which allow us to effectively "omit" solving Bethe equations (1.7). Performing similar computation for the diagonal components we arrive at

$$\mathcal{A}_{ii} = e^{g(k_i)-ik_ix}\frac{\Omega_i \sin(\pi\nu_i)^2}{L^2}\sum_q \frac{e^{iqx-g(q)}}{\sin^2\frac{q-k_i}{2}}. \tag{A.15}$$

The sum can be evaluated in the same way as in Eq. (A.12):

$$\mathcal{A}_{ii} = \Omega_i\left(1+\frac{(x+ig'(k_i))(e^{2\pi i\nu(k_i)}-1)}{L}\right) + e^{g(k_i)-ik_ix}\frac{\Omega_i\sin(\pi\nu_i)^2}{2L}\int\limits_{-\pi}^{\pi}\frac{dq}{\pi}\frac{e^{-g(q)+iqx}}{\sin^2\frac{q+i0-k_i}{2}}. \tag{A.16}$$

Equivalently, using definition (A.14), we can present

$$\mathcal{A}_{ii} = \Omega_i\left(1+\frac{2\pi\nu'(k_i)}{L}\right) + e^{g(k_i)-ik_ix}\frac{\Omega_i\sin(\pi\nu_i)^2}{2L}2E'(k_i). \tag{A.17}$$

So recalling definition of (A.5) we obtain for generic i and j

$$\mathcal{A}_{ij} = \delta_{ij} + \frac{\sin^2(\pi\nu_i)}{2L}e^{g(k_i)}e^{-i(k_i+k_j)x/2}e^{i(k_i-k_j)/2}\frac{E(k_i)-E(k_j)}{\sin\frac{k_i-k_j}{2}} + O(1/L^2), \tag{A.18}$$

where for $i=j$ the second term is understood in the L'Hopital rule sense. Similarly we obtain for the finite rank contribution

$$\delta\mathcal{A}_{ij} = -\frac{4}{L}\sin^2(\pi\nu_i)e^{g(k_i)}e^{-i(k_i+k_j)x/2} + O(1/L^2). \tag{A.19}$$

In this form we are at the position to take limit $L\to\infty$, and taking into account that k_i is quantized in the units $2\pi/L$, arrive at the Fredholm determinants (1.12).

Similarly, we can perform summation for $\tau_-(x)$ defined in Eq. (1.56). Instead of Eq. (A.6) we obtain the following

$$\tau_-(x) = \det(\mathcal{A}+\delta\mathcal{A}) + (\Gamma-1)\det\mathcal{A}, \tag{A.20}$$

where

$$\mathcal{A}_{ij} = \frac{1}{L^2} e^{-g(q_i)+ix(q_i+q_j)/2}$$
$$\times \sum_k \frac{e^{g(k)-ixk}\sin^2\pi\nu(k)}{1+\frac{2\pi}{L}\nu'(k)}\left(\cot\frac{k-q_i}{2}-i\right)\left(\cot\frac{k-q_j}{2}+i\right), \quad \text{(A.21)}$$

$$\delta A_{ij} = \frac{4}{L}F_+(q_i)F_-(q_i), \qquad \Gamma = -\frac{4}{L}\sum_k \frac{e^{g(k)-ixk}\sin^2\pi\nu(k)}{1+\frac{2\pi}{L}\nu'(k)}, \tag{A.22}$$

$$F_\pm(q) = e^{-\frac{g(q)}{2}+ix\frac{q}{2}}\frac{1}{L}\sum_k \frac{e^{g(k)-ixk}\sin^2\pi\nu(k)}{1+\frac{2\pi}{L}\nu'(k)}\left(\cot\frac{k-q}{2}\pm i\right). \tag{A.23}$$

Here $\sum\limits_k$ means sum over all $L+\delta$ nonequivalent (mod 2π) solutions of Eq. (1.7), which can be presented as a contour integral

$$\frac{1}{L}\sum_k \frac{f(k)}{1+\frac{2\pi\nu'(k)}{L}} = \oint_C \frac{dk}{2\pi}\frac{f(k)}{e^{ikL+2\pi i\nu(k)}-1}, \tag{A.24}$$

where the contour C runs around poles of the denominator only and avoids and singularities of $f(k)$. Then the derivation goes along the lines as for $\tau(x)$. Namely, for $i \neq j$ we present

$$\mathcal{A}_{ij} = \frac{1}{2L}e^{-g(q_i)+ix(q_i+q_j)/2}e^{i(q_i-q_j)/2}\frac{c(q_i)-c(q_j)}{\sin\frac{q_i-q_j}{2}}, \tag{A.25}$$

where now instead of Eq. (A.11)

$$c(q) = \frac{2}{L}\sum_k \frac{e^{g(k)-ikx}\sin^2(\pi\nu(k))}{1+\frac{2\pi\nu'(k)}{L}}\cot\frac{k-q}{2}. \tag{A.26}$$

In the thermodynamic limit this function can be replaced by $E(q)$, which does not depend on the system size

$$c(q) \approx E_-(q) = \frac{1}{\pi}\int\limits_{-\pi+i0}^{\pi+i0} dk\, e^{g(k)-ixk}\sin^2\pi\nu(k)\cot\frac{k-q}{2}$$
$$- 4i\frac{e^{g(q)-ixq}\sin^2\pi\nu(q)e^{-2\pi i\nu(q)}}{1-e^{-2\pi i\nu(q)}}. \tag{A.27}$$

For positive x it is much more convenient to rewrite this function as

$$E_-(q) = \frac{1}{\pi} \int\limits_{-\pi-i0}^{\pi-i0} dk\, e^{g(k)-ixk} \sin^2 \pi\nu(k) \cot \frac{k-q}{2}$$
$$- 4ie^{g(q)-ixq} \sin^2 \pi\nu(q) \left(1 + \frac{e^{-2\pi i\nu(q)}}{1-e^{-2\pi i\nu(q)}}\right). \quad (A.28)$$

Now if we relate

$$e^{-g(q)} = e^{2\pi i\nu(q)} - 1, \quad (A.29)$$

this function transform into

$$E_-(q) = \int\limits_{-\pi-i0}^{\pi-i0} \frac{dk}{4\pi}\, e^{-ixk}(e^{-2\pi i\nu(k)} - 1) \cot \frac{k-q}{2} + ie^{-ixq}. \quad (A.30)$$

For large positive x the integral can be neglected. For diagonal components we obtain

$$A_{ii} = 1 + \frac{1}{L} e^{-g(q_i)+ixq_i} E'_-(q_i). \quad (A.31)$$

Function Γ can be written as

$$\Gamma = \int\limits_{-\pi}^{\pi} \frac{dk}{2\pi} e^{-ixk}(1 - e^{-2\pi i\nu(k)}). \quad (A.32)$$

It is also exponentially suppressed for $x \to +\infty$. The finite rank contribution is easily evaluated taking into account that

$$F_\pm(q) = \frac{e^{-g(q)/2+ixq/2}}{2} \left(E_-(q) \mp i\Gamma/2\right). \quad (A.33)$$

After all these transformations one readily obtains the result Eq. (1.58) in the thermodynamic limit.

Appendix B

Lemmas about products

In this appendix, we study products that appear in the overlaps. We assume that $\nu(q)$ is a smooth function on the segment $[-\pi, \pi]$ and assign its values in specific points as ν_j, namely

$$\nu_j = \nu(q_j), \qquad q_j = \frac{2\pi}{L}\left(-\frac{L+1}{2} + j\right), \tag{B.1}$$

$$\nu_- = \nu(-\pi), \qquad \nu_+ = \nu(\pi), \qquad \delta \equiv \nu_+ - \nu_-. \tag{B.2}$$

First we consider constant function $\nu(q) = \nu = \text{const}$.

Lemma B.1. *The following product formula is valid*

$$B_L(\nu) \equiv \prod_{j=1}^{L-1} \frac{\sin\frac{\pi(j-\nu)}{L}}{\sin\frac{\pi j}{L}} = \frac{\sin(\pi\nu)}{L\sin\frac{\pi\nu}{L}}. \tag{B.3}$$

In the limit $L \to \infty$ this product simplifies to

$$B_L(\nu) \approx \frac{\sin(\pi\nu)}{\pi\nu}. \tag{B.4}$$

The denominator is equal to

$$\prod_{j=1}^{L-1} \sin\frac{\pi j}{L} = \frac{L}{2^{L-1}}. \tag{B.5}$$

Proof. We can rewrite identically the left hand side as

$$B_L(\nu) = \prod_{j=1}^{L-1} \frac{\sin\frac{\pi(j-\nu)}{L}}{\sin\frac{\pi j}{L}} = e^{-i\pi\nu\frac{L-1}{L}} \prod_{j=1}^{L-1} \frac{e^{\frac{2\pi i}{L}j} - e^{\frac{2\pi i\nu}{L}}}{e^{\frac{2\pi i}{L}j} - 1}. \tag{B.6}$$

Taking into account that

$$\prod_{j=1}^{L-1}\left(z - e^{2\pi ij/L}\right) = \frac{z^L - 1}{z - 1}, \tag{B.7}$$

we obtain

$$B_L(\nu) = \frac{\sin(\pi\nu)}{L\sin\frac{\pi\nu}{L}}. \tag{B.8}$$

$\square$

Further, we proceed with the generic function $\nu(q)$.

Lemma B.2. *For an integer $0 \le A \le L-1$, the following asymptotic approximation in the limit $L \to \infty$ is valid*

$$B_{A,L}[\nu(q)] \equiv \prod_{j=1}^{A} \frac{\sin\frac{\pi(j-\nu_j)}{L}}{\sin\frac{\pi j}{L}}$$
$$\approx L^{\nu_A}\,\Gamma\left[\begin{array}{c} A+1-\nu_1,\ L-A \\ A+1,\ 1-\nu_1,\ L-A+\nu_A \end{array}\right] \exp\int_{-\pi}^{q_A} f(q)\,dq\ , \quad \text{(B.9)}$$

where

$$\Gamma\left[\begin{array}{cccc} a_1, & a_2, & \dots & a_p \\ b_1, & b_2, & \dots & b_q \end{array}\right] = \frac{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_p)}{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_q)}, \qquad \text{(B.10)}$$

$$f(q) = -\frac{\nu(q_A)}{\pi - q} + \frac{\nu_-}{\pi+q} + \frac{\nu(q)}{2}\tan\frac{q}{2}. \qquad \text{(B.11)}$$

Proof. First, we introduce the modified product

$$\tilde{B}_{A,L}[\nu(q)] = \prod_{j=1}^{A} \frac{\sin\frac{\pi(j-\nu_j)}{L}}{\sin\frac{\pi j}{L}} \frac{1}{1-\frac{\nu_j}{j}} \frac{1}{1+\frac{\nu_j}{L-j}} = \prod_{j=1}^{A} \frac{\Gamma(1+\frac{j}{L})}{\Gamma(1+\frac{j-\nu_j}{L})} \frac{\Gamma(2-\frac{j}{L})}{\Gamma(2-\frac{j-\nu_j}{L})}. \qquad \text{(B.12)}$$

Due to this modification, it is enough to expand $\log\tilde{B}_{A,L}[\nu(q)]$ up to the linear terms in ν_j since higher orders will be of order $O(1/L)$, namely

$$\log\tilde{B}_{A,L}[\nu(q)] = \sum_{j=1}^{A} \nu_j \left(\frac{1}{j} - \frac{1}{L-j} - \frac{1}{L}\cot\frac{\pi j}{L}\right) + O(1/L). \qquad \text{(B.13)}$$

Taking into account (B.1) we transform the sum into an integral

$$\log\tilde{B}_{A,L}[\nu(q)] = \int_{-\pi}^{q_A} \nu(q)\left(\frac{1}{q+\pi} - \frac{1}{\pi-q} + \frac{1}{2\pi}\tan\frac{q}{2}\right) dq. \qquad \text{(B.14)}$$

The rest of the product can be evaluated in a similar manner. First, we identically transform

$$\prod_{j=1}^{A}\left(1-\frac{\nu_j}{j}\right)\left(1+\frac{\nu_j}{L-j}\right)$$
$$= \Gamma\left[\begin{array}{c} A+1-\nu_1,\ L+\nu_A,\ L-A \\ A+1,\ 1-\nu_1,\ L,\ L-A+\nu_A \end{array}\right] \prod_{j=1}^{A} \frac{\left(1-\frac{\nu_j}{j}\right)\left(1+\frac{\nu_j}{L-j}\right)}{\left(1-\frac{\nu_1}{j}\right)\left(1+\frac{\nu_A}{L-j}\right)} \qquad \text{(B.15)}$$

The logarithm of the remaining product can be expanded only up to linear in ν terms to capture finite terms in $L \to \infty$ limit, namely

$$\log \prod_{j=1}^{A} \frac{\left(1-\frac{\nu_j}{j}\right)\left(1+\frac{\nu_j}{L-j}\right)}{\left(1-\frac{\nu_1}{j}\right)\left(1+\frac{\nu_A}{L-j}\right)} = -\int_{-\pi}^{q_A} dq \frac{\nu(q)-\nu(-\pi)}{q+\pi} + \int_{-\pi}^{q_A} dq \frac{\nu(q)-\nu_A}{\pi-q} \tag{B.16}$$

Combining this result with (B.14) and using Stirling's formula we obtain the desired result (B.9). □

Remark 1. For $A = L-1$, using Stirling's approximation for Gamma functions we obtain

$$\prod_{j=1}^{L-1} \frac{\sin \frac{\pi(j-\nu_j)}{L}}{\sin \frac{\pi j}{L}} \approx \frac{L^{\nu_+-\nu_-}}{\Gamma(1+\nu_+)\Gamma(1-\nu_-)}$$

$$\times \exp \int_{-\pi}^{\pi} dq \left(\frac{\pi(\nu_+-\nu_-)+q(\nu_++\nu_-)}{q^2-\pi^2} + \frac{\nu(q)}{2} \tan \frac{q}{2} \right). \tag{B.17}$$

. **Remark 2.** For $A \sim L$ and $L - A \sim L$ the prefactor can be simplified as

$$\Gamma \left[\begin{array}{c} A+1-\nu_1,\ L+\nu_A,\ L-A \\ A+1,\ 1-\nu_1,\ L,\ L-A+\nu_A \end{array} \right]$$

$$= \frac{1}{L^{\nu_1}} \frac{1}{\Gamma(1-\nu_1)} \frac{1}{(A/L)^{\nu_1}(1-A/L)^{\nu_A}}. \tag{B.18}$$

The next lemma is a simple corollary of the previous one.

Lemma B.3. *The following asymptotic expression is valid as* $L \to \infty$

$$\mathcal{Z}_a \equiv \sin^2 \frac{\pi \nu_a}{L} \prod_{j\neq a}^{L} \frac{\sin^2 \frac{\pi(j-a-\nu_j)}{L}}{\sin^2 \frac{\pi(j-a)}{L}} \approx L^{2\delta-2} \sin^2(\pi\nu_a)$$

$$\times \Gamma \left[\begin{array}{c} a+\nu_a,\ L-a+1-\nu_a \\ a+\nu_+,\ L-a+1-\nu_- \end{array} \right]^2 e^{2F(q_a)}, \tag{B.19}$$

where

$$F(q_a) = \int_{q_a}^{\pi} \left(-\frac{\nu_+}{2\pi+q_a-q} + \frac{\nu_a}{q-q_a} - \frac{\nu(q)}{2} \cot \frac{q-q_a}{2} \right) dq$$

$$+ \int_{-\pi}^{q_a} \left(\frac{\nu_-}{2\pi+q-q_a} + \frac{\nu_a}{q-q_a} - \frac{\nu(q)}{2} \cot \frac{q-q_a}{2} \right) dq. \tag{B.20}$$

Proof. First, we identically present this product as

$$\mathcal{Z}_a = \sin^2 \frac{\pi\nu_a}{L} \prod_{j=1}^{a-1} \frac{\sin^2 \frac{\pi(j+\nu_{a-j})}{L}}{\sin^2 \frac{\pi j}{L}} \prod_{j=1}^{L-a} \frac{\sin^2 \frac{\pi(j-\nu_{j+a})}{L}}{\sin^2 \frac{\pi j}{L}}. \tag{B.21}$$

Then using Lemma (B.2) and Stirling's formula we obtain

$$\mathcal{Z}_a \approx L^{2\delta-2} \sin^2(\pi\nu_a)\Gamma \left[\begin{array}{c} a+\nu_a,\, L-a+1-\nu_a \\ a+\nu_+,\, L-a+1-\nu_- \end{array} \right]^2 e^{2F(q_a)} \tag{B.22}$$

with

$$\begin{aligned} F(q_a) = &\int_{-\pi}^{-q_a} dq \left(-\frac{\nu_+}{\pi - q} + \frac{\nu(q_a)}{\pi + q} + \frac{\nu(q_a + q + \pi)}{2} \tan \frac{q}{2} \right) - \\ &- \int_{-\pi}^{q_a} dq \left(-\frac{\nu_-}{\pi - q} + \frac{\nu(q_a)}{\pi + q} + \frac{\nu(q_a - q - \pi)}{2} \tan \frac{q}{2} \right). \end{aligned} \tag{B.23}$$

Changing variables we obtain the desired statement. □

Further, we proceed with double products.

Lemma B.4. *For $\delta \geq 0$ the following asymptotic expansion is valid in the limit $L \to \infty$*

$$Z \equiv \prod_{i=1}^{L} \prod_{j=1}^{i-1} \frac{\sin \frac{\pi}{L}(i-j-\nu_i+\nu_j)}{\sin \frac{\pi(i-j)}{L}} \approx \frac{\mathcal{A}}{L^{\delta^2/2}}, \tag{B.24}$$

where the L independent prefactor $\mathcal{A}$ reads

$$\begin{aligned} \mathcal{A} = &G(1+\delta)(2\pi)^{-\frac{\delta(\delta+1)}{2}} \exp\left(\frac{\delta}{2} - \delta F(\pi)\right) \\ &\times \left(-\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dk \left[\frac{\nu(q) - \nu(k) - \delta(q-k)/(2\pi)}{4 \sin \frac{q-k}{2}} \right]^2 \right) \end{aligned} \tag{B.25}$$

with $F(\pi)$ is defined in Eq. (B.20) *and $G(x)$ stands for Barnes G-function defined by the functional relation $G(x+1) = \Gamma(x)G(x)$. Notice that function $\nu(q)-\delta q/2\pi$ has zero winding number so the integrals in the exponential are well defined.*

Proof. To find the thermodynamic limit of Z we rewrite it as $Z = Y_1 Y_2 e^{R_\delta}$ with

$$\begin{aligned} Y_1 = &\prod_{i=1}^{L} \prod_{j=1}^{i-1} \frac{\sin \frac{\pi}{L}(i-j-\nu_i+\nu_j)}{\sin \frac{\pi(i-j)}{L}} \frac{1 - \frac{i-j}{L}}{1 - \frac{i-j-(\nu_i-\nu_j)}{L}} e^{\frac{\nu_i-\nu_j}{L-i+j}} = \\ &= \prod_{i=1}^{L} \prod_{j=1}^{i-1} \frac{\cos \frac{\pi(\nu_i-\nu_j)}{L}}{1 + \frac{\nu_i-\nu_j}{L(1-\frac{i-j}{L})}} \left(1 - \frac{\tan \frac{\pi(\nu_i-\nu_j)}{L}}{\tan \frac{\pi(i-j)}{L}} \right) e^{\frac{\nu_i-\nu_j}{L-i+j}}, \end{aligned} \tag{B.26}$$

$$Y_2 = \prod_{i=1}^{L}\prod_{j=1}^{i-1}\left(1+\frac{\delta}{L-i+j}\right)e^{-\frac{\delta}{L-i+j}}$$
$$\times\prod_{i=1}^{L}\prod_{j=1}^{i-1}\left(1+\frac{\nu_i-\nu_j-\delta}{L-i+j+\delta}\right)e^{-\frac{\nu_i-\nu_j-\delta}{L-i+j+\delta}}, \quad \text{(B.27)}$$

$$R_\delta = \sum_{i=1}^{L}\sum_{j=1}^{i-1}\left(\frac{\nu_i-\nu_j-\delta}{L-i+j+\delta}+\frac{\delta}{L-i+j}-\frac{\nu_i-\nu_j}{L-i+j}\right). \quad \text{(B.28)}$$

The factors are designed in such a way that terms $O(\nu^n)$ for $n>2$ do not contribute in $L\to\infty$ case. In particular, we used that

$$\sum_{j=1}^{L-1}\cot\frac{\pi j}{L} = 0. \quad \text{(B.29)}$$

So keeping only quadratic terms we obtain

$$\log Y_1 = \sum_{i=1}^{L}\sum_{j=1}^{i-1}\frac{\pi^2}{2L^2}(\nu_i-\nu_j)^2\left(\frac{1}{\pi^2}\left(1-\frac{i-j}{L}\right)^{-2}-\frac{1}{\sin^2\frac{\pi(i-j)}{L}}\right) \quad \text{(B.30)}$$

and taking $L\to\infty$

$$\log Y_1 = \frac{1}{8}\int_{-\pi}^{\pi}dq\int_{-\pi}^{q}dk(\nu(q)-\nu(k))^2\left(\frac{4}{(2\pi-q+k)^2}-\frac{1}{\sin^2\frac{q-k}{2}}\right). \quad \text{(B.31)}$$

Similarly

$$\log\prod_{i=1}^{L}\prod_{j=1}^{i-1}\left(1+\frac{\nu_i-\nu_j-\delta}{L-i+j+\delta}\right)e^{-\frac{\nu_i-\nu_j-\delta}{L-i+j+\delta}}$$
$$\approx -\frac{1}{2}\int_{-\pi}^{\pi}dq\int_{-\pi}^{q}dk\left(\frac{\nu(q)-\nu(k)-\delta}{2\pi-q+k}\right)^2. \quad \text{(B.32)}$$

The first part of the product in Y_2 (by grouping terms with the same $i-j$) can be presented as

$$W(\delta)\equiv\prod_{i=1}^{L}\prod_{j=1}^{i-1}\left(1+\frac{\delta}{L-i+j}\right)e^{-\frac{\delta}{L-i+j}} = \prod_{j=1}^{L-1}\left[\left(1+\frac{\delta}{j}\right)^j e^{-\delta}\right]. \quad \text{(B.33)}$$

We consider an additional expression

$$W_0(\delta)\equiv\prod_{j=1}^{L-1}\left(1+\frac{\delta}{j}\right) = \frac{\Gamma(L+\delta)}{\Gamma(1+\delta)\Gamma(L)}. \quad \text{(B.34)}$$

Differentiating it by δ we obtain

$$\frac{d\log W(\delta)}{d\delta} = -\delta\frac{d\log W_0(\delta)}{d\delta}. \tag{B.35}$$

For large L we can approximate

$$W_0(\delta) \approx \frac{L^\delta}{\Gamma(1+\delta)}. \tag{B.36}$$

Solving Eq. (B.35) with initial condition $\log W(\delta=0)=0$ we obtain

$$\log W(\delta) \approx -\frac{\delta^2}{2}\log L + \int_0^\delta z\frac{d\log\Gamma(1+z)}{dz}dz. \tag{B.37}$$

Finally, let us find $L\to\infty$ expression for R_δ defined in Eq. (B.28). First we identically transform it into

$$R_\delta = \sum_{i=1}^{L}(\nu_i-\nu_L)S_{L-i+1} - \sum_{i=1}^{L}(\nu_i-\nu_1)S_i. \tag{B.38}$$

$$S_i = \sum_{j=i}^{L-1}\left(\frac{1}{j+\delta}-\frac{1}{j}\right) = \frac{d}{d\epsilon}\log\frac{\Gamma(L+\epsilon+\delta)\Gamma(i+\epsilon)}{\Gamma(L+\epsilon)\Gamma(i+\epsilon+\delta)}\Big|_{\epsilon=0}. \tag{B.39}$$

From the form of Eq. (B.38) one can conclude that as $L\to\infty$ the non-vanishing contributions to the sum will come from indices $i=O(L)$. Theqrefore, using Stirling's formula we can present S_i as

$$S_i \approx \frac{d}{d\epsilon}\log\left(\frac{L+\epsilon}{i+\epsilon}\right)^\delta\Big|_{\epsilon=0} = \delta\left(\frac{1}{L}-\frac{1}{i}\right). \tag{B.40}$$

Theqrefore $R_\delta \approx \delta R$ with

$$\begin{aligned}R = \lim_{L\to\infty}&\left(\sum_{i=1}^{L}(\nu_i-\nu_L)\left(\frac{1}{L}-\frac{1}{L-i+1}\right)-\sum_{i=1}^{L-1}(\nu_i-\nu_1)\left(\frac{1}{L}-\frac{1}{i}\right)\right)\\ &= \int_{-\pi}^{\pi}dq(\nu(q)-\nu(\pi))\left(\frac{1}{2\pi}-\frac{1}{\pi-q}\right)\\ &\qquad - \int_{-\pi}^{\pi}dq(\nu(q)-\nu(-\pi))\left(\frac{1}{2\pi}-\frac{1}{\pi+q}\right).\end{aligned} \tag{B.41}$$

So far we have proved that

$$Z \approx L^{-\delta^2/2} e^{C_\delta}. \tag{B.42}$$

with

$$C_\delta = \delta R + \int_0^\delta z \frac{d \log \Gamma(1+z)}{dz} dz + \frac{1}{2} \int_{-\pi}^{\pi} dq \int_{-\pi}^{q} dk \left(\frac{2\delta(\nu(q) - \nu(k)) - \delta^2}{(2\pi - q + k)^2} - \frac{(\nu(q) - \nu(k))^2}{4 \sin^2 \frac{q-k}{2}} \right). \tag{B.43}$$

Further, we can use

$$\int_0^\delta z \frac{d \log \Gamma(1+z)}{dz} dz = \frac{\delta(\delta+1)}{2} - \frac{\delta}{2} \log(2\pi) + \log G(1+\delta), \tag{B.44}$$

where $G(x)$ is Barnes G-function. The final answer is obtained by tedious but straightforward manipulations with integrals. $\square$

In the next lemma we address a similar double product for negative winding numbers $\delta < 0$.

Lemma B.5. *Let us define $\ell = L + \delta$, with $\delta < 0$, then the following asymptotic behavior is valid as $L \to \infty$ (here we still assume that $|\delta| \ll L$)*

$$Z \equiv \prod_{i=1}^{\ell} \prod_{j=1}^{i-1} \frac{\sin \frac{\pi}{L}(i - j - \nu_i + \nu_j)}{\sin \frac{\pi(i-j)}{L}} \approx \frac{L^{\delta^2/2} (2\pi)^{-(\delta^2+\delta)/2} e^{\delta/2}}{G(1-\delta)} \times \exp\left(- \int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dk \left[\frac{\nu(q) - \nu(k) - \delta(q-k)/(2\pi)}{4 \sin \frac{q-k}{2}} \right]^2 \right). \tag{B.45}$$

Proof. We present this product as a ratio $Z = Z_1/Z_2$ with

$$Z_1 = \prod_{i=1}^{\ell} \prod_{j=1}^{i-1} \frac{\sin \frac{\pi}{L}(i - j - \nu_i + \nu_j)}{\sin \frac{\pi(i-j)}{\ell}}, \qquad Z_2 = \prod_{i=1}^{\ell} \prod_{j=1}^{i-1} \frac{\sin \frac{\pi(i-j)}{L}}{\sin \frac{\pi(i-j)}{\ell}}. \tag{B.46}$$

We can identically transform Z_1 as

$$Z_1 = \prod_{i=1}^{\ell} \prod_{j=1}^{i-1} \frac{\sin \frac{\pi}{\ell}(i - j - [\nu_\delta]_i + [\nu_\delta]_j)}{\sin \frac{\pi(i-j)}{\ell}}, \tag{B.47}$$

where $[\nu_\delta]_i = \nu_i(1+\delta/L) - \delta i/L$. In thermodynamic limit this expression correspond to the following function

$$\nu_\delta(q) = \nu(q) - \delta\frac{\pi+q}{2\pi}. \tag{B.48}$$

This function has zero winding number, so applying the previous lemma, we obtain

$$Z_1 = \exp\left(-\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dk \left[\frac{\nu(q)-\nu(k)-\delta(q-k)/(2\pi)}{4\sin\frac{q-k}{2}}\right]^2\right). \tag{B.49}$$

Similarly, we can evaluate Z_2. We present it as

$$Z_2 = \prod_{i=1}^{\ell}\prod_{j=1}^{i-1} \frac{\sin\frac{\pi}{\ell}\left(i-j+\frac{\delta(i-j)}{L}\right)}{\sin\frac{\pi(i-j)}{\ell}}. \tag{B.50}$$

This corresponds to the positive phase shift $\nu(q) = -\delta q/(2\pi) = |\delta| q/(2\pi)$, and allows us to use previous lemma once again and obtain

$$Z_2 = \frac{G(1-\delta)(2\pi)^{(\delta^2+\delta)/2} e^{-\delta/2}}{\ell^{\delta^2/2}}. \tag{B.51}$$

$\square$

Here we used that $F(\pi) = -|\delta|\log(2\pi)$ for $\nu(q) = |\delta|q/(2\pi)$ (see Eq. (B.20)). Finally, Eqs. (B.49) and (B.51) immediately lead to the statement of the lemma.

Appendix C

Orthogonality catastrophe on the lattice

Here using results from Appendix B we evaluate the overlaps in Eq. (1.10).

C.1 Winding number $\delta = 1$

For $\delta = 1$ there exist $L+1$ solutions of Eq. (1.7)

$$k_j = \frac{2\pi}{L}\left(-\frac{L+1}{2} + j - \nu_j\right), \qquad \nu_j = \nu(k_j), \qquad j = 1, 2, \ldots, L+1. \tag{C.1}$$

We use all of them in Eq. (1.10) and set $\mathbf{q} = \{q_1, \dots q_L\}$ with

$$q_j = \frac{2\pi}{L}\left(-\frac{L+1}{2} + j\right), \qquad j = 1, 2, \dots, L. \tag{C.2}$$

To evaluate Eq. (1.10) in thermodynamic limit $L \to \infty$, we first we transform identically the determinant (1.11) as

$$(\det D)^2 = \prod_{i>j}^{L} \frac{\sin^2 \frac{k_i - k_j}{2}}{\sin^2 \frac{q_i - q_j}{2}} \times \prod_{j=1}^{L} \frac{\sin^2 \frac{k_{L+1} - k_j}{2}}{\sin^2 \frac{k_{L+1} - q_j}{2}} \times \prod_{i=1}^{L} \frac{\prod_{j\neq i}^{L} \sin^2 \frac{q_i - q_j}{2}}{\prod_{j=1}^{L} \sin^2 \frac{k_i - q_j}{2}}. \tag{C.3}$$

We analyze this expression term by term. The last product can be written down using Eqs. (1.21) and (1.22) as

$$\frac{\prod_{j\neq i}^{L} \sin^2 \frac{q_i - q_j}{2}}{\prod_{j=1}^{L} \sin^2 \frac{k_i - q_j}{2}} = \frac{1}{\sin^2 \frac{\pi\nu_i}{L}} \prod_{j=1}^{L-1} \frac{\sin^2 \frac{\pi j}{L}}{\sin^2 \frac{\pi(j-\nu_i)}{L}} = \frac{L^2}{\sin^2 \pi\nu_i}. \tag{C.4}$$

In the last step, we used Lemma (B.1). The next product can be evaluated employing similar transformations and using Lemma (B.2), namely

$$\begin{aligned}\prod_{j=1}^{L} \frac{\sin^2 \frac{k_{L+1} - k_j}{2}}{\sin^2 \frac{k_{L+1} - q_j}{2}} &= \frac{\sin^2 \frac{\pi\delta}{L}}{\sin^2 \frac{\pi\nu_+}{L}} \prod_{j=1}^{L-1} \frac{\sin^2 \frac{\pi}{L}\left(j - \nu_+ + \nu_{L+1-j}\right)}{\sin^2 \frac{\pi j}{L}} \prod_{j=1}^{L-1} \frac{\sin^2 \frac{\pi j}{L}}{\sin^2 \frac{\pi(j-\nu_+)}{L}} \\ &\approx \frac{\pi^2 L^2}{\sin^2 \pi\nu_+} \exp\left(\int_{-\pi}^{\pi} dq f_1(q)\right),\end{aligned} \tag{C.5}$$

where $\nu_+ = \nu_{L+1}$ and $\delta = \nu_+ - \nu_1 = \nu(\pi) - \nu(-\pi) = 1$ and

$$f_1(q) = \frac{2}{q - \pi} + (\nu(\pi) - \nu(-q)) \tan \frac{q}{2}. \tag{C.6}$$

Notice that

$$\int_{-\pi}^{\pi} dq f_1(q) = 2F(\pi) = 2F(-\pi) \tag{C.7}$$

with $F(q)$ defined in Eq. (B.20). Contrary to the expression (C.4), Eq. (C.5) is asymptotic as $L \to \infty$. Finally, the first double product in Eq. (C.3) can be evaluated using Lemma (B.4).

$$\prod_{i>j}^{L} \frac{\sin^2 \frac{k_i - k_j}{2}}{\sin^2 \frac{q_i - q_j}{2}} = \prod_{i=1}^{L} \prod_{j=1}^{i-1} \frac{\sin^2 \frac{\pi}{L}(i - j - \nu_i + \nu_j)}{\sin^2 \frac{\pi(i-j)}{L}} \approx \frac{\mathcal{A}^2}{L}. \tag{C.8}$$

Where A is defined in Eq. (B.25). The rest of the product in Eq. (1.10) can be evaluated for generic δ

$$\prod_{i=1}^{L+1}\left(1+\frac{2\pi}{L}\nu'(k_i)\right)\approx\exp\left(\int_{-\pi}^{\pi}\nu'(q)dq\right)=e^{\delta}, \tag{C.9}$$

$$\begin{aligned}\prod_{i=1}^{L}e^{g(k_i)-g(q_i)}&\approx\exp\left(-\int_{-\pi}^{\pi}g'(q)\nu(q)dq\right)\\&=\exp\left(2\pi i\int_{-\pi}^{\pi}\frac{\nu(q)\nu'(q)}{e^{2\pi i\nu(q)}-1}dq\right)=\left(1-e^{-2\pi i\nu_+}\right)^{\delta},\end{aligned} \tag{C.10}$$

where in the last part we have used the relation between $g(q)$ and $\nu(q)$ Eq. (1.16) and assumed that $\nu(k)$ has a non-vanishing imaginary part. Combining all factors together in Eq. (1.10) we obtain

$$\begin{aligned}|\langle\mathbf{k}|\mathbf{q}\rangle|^2&=4\pi^2\mathcal{A}^2e^{2F(\pi)-1}\\&=\exp\left(-\frac{1}{2}\int_{-\pi}^{\pi}dq\int_{-\pi}^{\pi}dk\left[\frac{\nu(q)-\nu(k)-(q-k)/2\pi}{2\sin\frac{q-k}{2}}\right]^2\right).\end{aligned} \tag{C.11}$$

C.2 Winding number $\delta=2$

For $\delta>1$ computation of the overlaps goes in the similar manner as in the previous section. Namely, first we consider overlap with the set $\tilde{\mathbf{k}}=k_1,\dots k_{L+1}$ with k_j defined in Eq. (C.1). There instead of Eq. (C.5) we will have

$$\prod_{j=1}^{L}\frac{\sin^2\frac{k_{L+1}-k_j}{2}}{\sin^2\frac{k_{L+1}-q_j}{2}}=\frac{L^2}{\sin^2(\pi\nu_+)}\frac{\pi^2L^{2\delta-2}}{\Gamma(\delta)^2}e^{2F(\pi)}, \tag{C.12}$$

with $F(\pi)$ defined in Eq. (B.20). Further, Eq. (C.8) we will replaced accordingly to Lemma (B.4)

$$\prod_{i>j}^{L}\frac{\sin^2\frac{k_i-k_j}{2}}{\sin^2\frac{q_i-q_j}{2}}=\prod_{i=1}^{L}\prod_{j=1}^{i-1}\frac{\sin^2\frac{\pi}{L}(i-j-\nu_i+\nu_j)}{\sin^2\frac{\pi(i-j)}{L}}\approx\frac{\mathcal{A}^2}{L^{\delta^2}}. \tag{C.13}$$

Taking into account Eqs. (C.9) and (C.10), for the corresponding function $g(k)$ (see Eq. (1.57)) we find the thermodynamic form for the overlap

$$
\begin{aligned}
|\langle\tilde{\mathbf{k}}|\mathbf{q}\rangle|^2 = {} & \frac{G(\delta)^2}{L^{(\delta-1)^2}}\frac{(1-e^{2\pi i\nu_+})^{\delta-1}}{(2\pi)^{(\delta-1)(\delta+2)}}e^{-2F(\pi)(\delta-1)} \\
& \times \exp\left(-\frac{1}{2}\int_{-\pi}^{\pi}dq\int_{-\pi}^{\pi}dk\left[\frac{\nu(q)-\nu(k)-\delta(q-k)/2\pi}{2\sin\frac{q-k}{2}}\right]^2\right). \qquad \text{(C.14)}
\end{aligned}
$$

The overlaps for other sets $\mathbf{k}$ can be obtained from this one. We further focus on $\delta = 2$, in this case there are exacly $L+2$ sets $\mathbf{k}$ parametrized by the omission of one of the solutions of Eq. (1.7), namely

$$
\mathbf{k}^{(a)} = \{k_1, \dots, k_{a-1}, k_{a+1}, \dots, k_{L+2}\}, \qquad a = 1, 2, \dots, L+2. \qquad \text{(C.15)}
$$

With this notations $\mathbf{k}^{(L+2)} = \tilde{\mathbf{k}}$. Now let us consider ratio of the excited overlap

$$
\begin{aligned}
\frac{|\langle\mathbf{k}^{(a)}|\mathbf{q}\rangle|^2}{|\langle\tilde{\mathbf{k}}|\mathbf{q}\rangle|^2} &= e^{g(\pi)-g(k_a)}\frac{\prod_{j=1}^{L+1}\sin^2\frac{k_{L+2}-k_j}{2}}{\prod_{j\neq a}^{L+2}\sin^2\frac{k_a-k_j}{2}} \\
&= e^{g(\pi)-g(k_a)}\frac{\sin^2\frac{\pi}{L}\sin^2\frac{2\pi}{L}\prod_{j=1}^{L-1}\sin^2\frac{\pi}{L}(j-\nu_{j+2}+\nu_+)}{\prod_{j=1}^{a-1}\sin^2\frac{\pi(j-\nu_a+\nu_{a-j})}{L}\prod_{j=1}^{L+2-a}\sin^2\frac{\pi(j-\nu_{j+a}+\nu_a)}{L}}. \qquad \text{(C.16)}
\end{aligned}
$$

Using Lemma (B.2) for $a \sim L$, $L - a \sim L$ we obtain

$$
\frac{|\langle\mathbf{k}^{(a)}|\mathbf{q}\rangle|^2}{|\langle\tilde{\mathbf{k}}|\mathbf{q}\rangle|^2} = (2\pi)^4\exp\left[2F(\pi) + \fint_{-\pi}^{\pi}\left(\nu(q)-\frac{q}{\pi}\right)\cot\frac{q-k_a}{2}dq\right]. \qquad \text{(C.17)}
$$

Combining this result with Eq. (C.14) for $\delta = 2$, the overlap can be written as

$$
\begin{aligned}
|\langle\mathbf{k}^{(a)}|\mathbf{q}\rangle|^2 = {} & -\frac{e^{2\pi i\nu(k)}-1}{L}\exp\left[\fint_{-\pi}^{\pi}\left(\nu(q)-\frac{q}{\pi}\right)\cot\frac{q-k_a}{2}dq\right] \\
& \times\left[-\frac{1}{2}\int_{-\pi}^{\pi}dq\int_{-\pi}^{\pi}dk\left(\frac{\nu(q)-\nu(k)-(q-k)/\pi}{2\sin\frac{q-k}{2}}\right)^2\right]. \qquad \text{(C.18)}
\end{aligned}
$$

C.3 Winding number $\delta = 0$

Let us study thermodynamic limit of the overlap (1.10) in the case $N = L - 1$, which is especially useful for $\delta = 0$. Below, however, for the sake of generality, we will keep $\delta \geq 0$. Our goal is to evaluate Z_a defined via

$$|\langle \mathbf{k}|\mathbf{q}^{(a)}\rangle|^2 \equiv e^{g(q_a)} Z_a. \tag{C.19}$$

We use notations (1.22) and (1.21) to label the momenta and Eq. (1.25) for $\mathbf{q}^{(a)}$. Let $\det D^{(a)}$ be the determinant in Eq. (1.11) that corresponds to the set $\mathbf{q}^{(a)}$. It explicitly reads as

$$\det D^{(a)} = \frac{\prod\limits_{i>j}^{L} \sin \frac{k_i - k_j}{2} \prod\limits_{\substack{i>j \\ i,j\neq a}}^{L} \sin \frac{q_j - q_i}{2}}{\prod\limits_{i=1}^{L} \prod\limits_{\substack{j=1 \\ j\neq a}}^{L} \sin \frac{k_i - q_j}{2}}. \tag{C.20}$$

We can present it identically as

$$\begin{aligned} \prod_{i=1}^{L} \left(\frac{\sin \pi\nu_i}{L} \right)^2 (\det D^{(a)})^2 &= \prod_{i=1}^{L} \prod_{j=1}^{i-1} \frac{\sin^2 \frac{k_i - k_j}{2}}{\sin^2 \frac{q_i - q_j}{2}} \\ &\times \prod_{i=1}^{L} \left(\frac{\sin \pi\nu_i}{L} \right)^2 \frac{\prod_{j\neq i}^{L} \sin^2 \frac{q_i - q_j}{2}}{\prod\limits_{j=1}^{L} \sin^2 \frac{k_i - q_j}{2}} \times \sin^2 \frac{\pi\nu_a}{L} \prod_{j\neq a} \frac{\sin^2 \frac{k_j - q_a}{2}}{\sin^2 \frac{q_j - q_a}{2}} \end{aligned} \tag{C.21}$$

The last part of this product is nothing but $\mathcal{Z}_a$ in Eq. (B.19), the middle part is equal to 1 due to to Lemma (B.1), while the first part can evaluated with Lemma (B.4) and gives $\mathcal{A}^2/L^{\delta^2}$. Overall we have[1]

$$\begin{aligned} \prod_{i=1}^{L} \left(\frac{\sin \pi\nu_i}{L} \right)^2 (\det D^{(a)})^2 &\approx \frac{\mathcal{A}^2 e^{2F(q_a)}}{L^{\delta^2 - 2\delta + 2}} \sin^2(\pi\nu_a) \\ &\times \left[\frac{\Gamma(L - a + 1 - \nu_a)\Gamma(a + \nu_a)}{\Gamma(L - a + 1 - \nu_-)\Gamma(a + \nu_+)} \right]^2, \end{aligned} \tag{C.22}$$

where $F(q_a)$ is given by Eq. (B.20).

[1] Recall that $\nu_a \equiv \nu(q_a)$.

Taking into account Eqs. (C.9) and (C.10), we obtain

$$Z_a = -4(1 - e^{-2\pi i \nu_+})^{\delta} \frac{\mathcal{A}^2 e^{2F(q_a)-\delta}}{L^{(\delta-1)^2}} \sin^2(\pi\nu_a) \times \left[\frac{\Gamma(L-a+1-\nu_a)\Gamma(a+\nu_a)}{\Gamma(L-a+1-\nu_-)\Gamma(a+\nu_+)}\right]^2. \quad \text{(C.23)}$$

For $\delta = 0$ we can rewrite this expression as

$$Z_a = \frac{A[q_a]}{L} \left[\frac{\Gamma(L-a+1-\nu_a)\Gamma(a+\nu_a)}{\Gamma(L-a+1-\nu_+)\Gamma(a+\nu_+)}\right]^2 \left(\frac{\pi+q_a}{\pi-q_a}\right)^{2\nu_+-2\nu_a}, \quad \text{(C.24)}$$

$$A[q_a] = -4\sin^2(\pi\nu_a) \exp\left(-\fint_{-\pi}^{\pi} \nu(q) \cot\frac{q-q_a}{2} dq\right) \times \exp\left(-\frac{1}{2}\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dk \left[\frac{\nu(q)-\nu(k)}{2\sin\frac{q-k}{2}}\right]^2\right), \quad \text{(C.25)}$$

where the integral is understood as the principal value.

For $\delta = 1$ we can rewrite this expression as

$$Z_a = 4\sin^2(\pi\nu_a)(e^{-2\pi i\nu_+} - 1)\mathcal{A}^2 e^{2F(q_a)-1} \times \left[\frac{\Gamma(L-a+1-\nu_a)\Gamma(a+\nu_a)}{\Gamma(L-a+2-\nu_+)\Gamma(a+\nu_+)}\right]^2. \quad \text{(C.26)}$$

Using expression (C.7) and (B.25) we obtain

$$Z_a = \frac{\sin^2(\pi\nu_a)}{\pi^2}(e^{-2\pi i\nu_+} - 1)|\langle \mathbf{k}|\mathbf{q}\rangle|^2 e^{2F(q_a)-2F(\pi)} \times \left[\frac{\Gamma(L-a+1-\nu_a)\Gamma(a+\nu_a)}{\Gamma(L-a+2-\nu_+)\Gamma(a+\nu_+)}\right]^2 \quad \text{(C.27)}$$

with $|\langle \mathbf{k}|\mathbf{q}\rangle|^2$ given by Eq. (C.11).

C.4 Negative winding number $\delta < 0$

Following Sec. (1.3.3) we fix $\delta = 1 - n$ with $n \in \mathbb{Z}_{\geq}$, $\ell = L + \delta$, the set $\mathbf{k} = \{k_1, \dots k_\ell\}$ is given as

$$k_i = \frac{2\pi}{L}\left(-\frac{L+1}{2} + i - \nu_i\right), \qquad i = 1, 2, \dots \ell, \quad \text{(C.28)}$$

the set $\mathbf{q}^{a_1,\dots a_n}$ is obtained from the complete set $\mathbf{q}$ in Eq. (C.2) by the omission of the “particle” at position q_{a_i}

$$\mathbf{q}^{a_1,\dots a_n} = \{q_1, \dots \hat{q}_{a_1}, \dots \hat{q}_{a_n}, \dots q_L\}. \tag{C.29}$$

The determinant (1.11) in (1.10) after certain restructuring of the factors and employing Lemma (B.1) reads

$$\begin{aligned}
&\prod_{i=1}^{\ell}\left(\frac{\sin\pi\nu_i}{L}\right)^2 (\det D)^2 \\
&\qquad = \prod_{i=1}^{\ell} \frac{\prod\limits_{j=1}^{L} \sin^2\frac{k_i-q_j}{2}}{\prod_{j\neq i}^{L}\sin^2\frac{q_i-q_j}{2}} \times \frac{\prod\limits_{i>j}^{\ell}\sin^2\frac{k_i-k_j}{2} \prod\limits_{\substack{i>j\\ i,j\neq a_1,\dots,a_n}}^{L} \sin^2\frac{q_j-q_i}{2}}{\prod\limits_{i=1}^{\ell}\prod\limits_{\substack{j=1\\ j\neq a_1,\dots,a_n}}^{L}\sin^2\frac{k_i-q_j}{2}} \\
&= \frac{\prod\limits_{i>j}^{\ell}\sin^2\frac{k_i-k_j}{2}}{\prod\limits_{i>j}^{\ell}\sin^2\frac{q_i-q_j}{2}} \times \frac{\prod\limits_{i>j}^{\ell}\sin^2\frac{q_i-q_j}{2}\prod\limits_{i>j}^{L}\sin^2\frac{q_j-q_i}{2}}{\prod\limits_{i=1}^{\ell}\prod\limits_{\substack{j=1\\ j\neq i}}^{L}\sin^2\frac{q_i-q_j}{2}} \times \prod_{i>j}^{n}\sin^2\frac{q_{a_i}-q_{a_j}}{2} \times \prod_{i=1}^{n}\tilde{\mathcal{Z}}_{a_i}
\end{aligned} \tag{C.30}$$

with

$$\tilde{\mathcal{Z}}_a = \frac{\prod\limits_{i=1}^{\ell}\sin^2\frac{k_i-q_a}{2}}{\prod\limits_{i\neq a}^{L}\sin^2\frac{q_i-q_a}{2}}. \tag{C.31}$$

The first factor in this expression can be evaluated via Lemma (B.5)

$$\begin{aligned}
\frac{\prod\limits_{i>j}^{\ell}\sin^2\frac{k_i-k_j}{2}}{\prod\limits_{i>j}^{\ell}\sin^2\frac{q_i-q_j}{2}} &= \frac{L^{\delta^2}(2\pi)^{-(\delta^2+\delta)}e^{\delta}}{G(1-\delta)^2} \\
&\times \exp\left(-\frac{1}{2}\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dk \left[\frac{\nu(q)-\nu(k)-\delta(q-k)/(2\pi)}{2\sin\frac{q-k}{2}}\right]^2\right).
\end{aligned} \tag{C.32}$$

The second factor reads

$$\frac{\prod\limits_{i>j}^{\ell} \sin^2 \frac{q_i-q_j}{2} \prod\limits_{i>j}^{L} \sin^2 \frac{q_j-q_i}{2}}{\prod\limits_{i=1}^{\ell} \prod\limits_{\substack{j=1\\ j\neq i}}^{L} \sin^2 \frac{q_i-q_j}{2}} = \prod_{i>j>\ell}^{L} \sin^2 \frac{q_i-q_j}{2} = \prod_{i=1}^{n-1}\prod_{j=1}^{i-1} \sin^2 \frac{\pi(i-j)}{L}$$

$$\approx \left(\frac{\pi}{L}\right)^{(n-2)(n-1)} \prod_{i=1}^{n-1}\prod_{j=1}^{i-1} (i-j)^2 = \left(\frac{\pi}{L}\right)^{(n-2)(n-1)} \prod_{i=1}^{n-1}\prod_{j=1}^{i-1} j^2$$

$$= \left(\frac{\pi}{L}\right)^{(n-2)(n-1)} \prod_{i=1}^{n-1} \Gamma(i)^2 = \left(\frac{\pi}{L}\right)^{(n-2)(n-1)} G(n)^2$$

$$= \left(\frac{\pi}{L}\right)^{\delta(\delta+1)} G(1-\delta)^2. \quad \text{(C.33)}$$

We evaluate $\tilde{\mathcal{Z}}_a$ in Eq. (C.31) for $a \sim L$ and $L-a \sim L$. We complete $\tilde{\mathcal{Z}}_a$ to the full product $\mathcal{Z}_a$ in Eq. (B.19) and approximate it as

$$\tilde{\mathcal{Z}}_a = \frac{\mathcal{Z}_a}{\prod\limits_{j=\ell+1}^{L} \sin^2 \frac{\pi(j-a-\nu_j)}{L}} \approx \frac{\mathcal{Z}_a}{\left(\cos \frac{q_a}{2}\right)^{2|\delta|}}. \quad \text{(C.34)}$$

To approximate further $\mathcal{Z}_a$ in Eq. (B.19) we notice that

$$L^{\delta}\Gamma\left[\begin{array}{c} a+\nu_a,\, L-a+1-\nu_a \\ a+\nu_+,\, L-a+1-\nu_- \end{array}\right] \approx \left(\frac{a}{L}\right)^{\nu_a-\nu_+} \left(1-\frac{a}{L}\right)^{\nu_- -\nu_a}$$

$$= \left(\frac{\pi+q_a}{2\pi}\right)^{\nu_a-\nu_+} \left(\frac{\pi-q_a}{2\pi}\right)^{\nu_- -\nu_a}. \quad \text{(C.35)}$$

Further, we can simplify $F(q_a)$ using that in the principal value

$$\fint_{-\pi}^{\pi} dq q \cot \frac{q-q_a}{2} = 4\pi \log \left|2 \cos \frac{q_a}{2}\right|. \quad \text{(C.36)}$$

So thermodynamic limit for $\tilde{\mathcal{Z}}_a$ reads

$$\tilde{\mathcal{Z}}_a = 4^{|\delta|} \frac{\sin^2(\pi\nu_a)}{L^2} \exp\left[-\fint_{-\pi}^{\pi} dq \left(\nu(q) - \delta\frac{q}{2\pi}\right) \cot \frac{q-q_a}{2}\right]. \quad \text{(C.37)}$$

The remaining factors in Eq. (1.10) can be evaluated with the help of Eqs. (C.9) and (C.10)

$$\frac{\prod_{i=1}^{\ell} e^{g(k_i)} \prod_{q_i\in\mathbf{q}^{a_1,\dots a_n}} e^{-g(q_i)}}{\prod_{i=1}^{\ell}\left(1+\frac{2\pi}{L}\nu'(k_i)\right)} = \frac{\prod_{i=1}^{\ell} e^{g(k_i)-g(q_i)} \prod_{i=\ell+1}^{L} e^{-g(q_i)} \prod_{i=1}^{n} e^{g(q_{a_i})}}{\prod_{i=1}^{\ell}\left(1+\frac{2\pi}{L}\nu'(k_i)\right)}$$
$$= (-1)^{\delta} e^{-\delta} \prod_{i=1}^{n} e^{g(q_{a_i})}. \quad \text{(C.38)}$$

Finally, the overlap (1.10) in the thermodynamic limit can be written as

$$|\langle\mathbf{k}|\mathbf{q}^{a_1,\dots a_n}\rangle|^2 = \prod_{i>j}^{n}\left(2\sin\frac{q_{a_i}-q_{a_j}}{2}\right)^2 \prod_{i=1}^{n}\mathcal{Y}_{a_i}$$
$$\times \exp\left(-\frac{1}{2}\int_{-\pi}^{\pi}dq\int_{-\pi}^{\pi}dk\left[\frac{\nu(q)-\nu(k)-\delta(q-k)/(2\pi)}{2\sin\frac{q-k}{2}}\right]^2\right) \quad \text{(C.39)}$$

with

$$\mathcal{Y}_a = -4\frac{\sin^2(\pi\nu_a)}{L}\exp\left[g(q_a) - \fint_{-\pi}^{\pi} dq\left(\nu(q)-\delta\frac{q}{2\pi}\right)\cot\frac{q-q_a}{2}\right]. \quad \text{(C.40)}$$

C.5 Overlaps for τ_0

Now let us consider how overlaps defined in Eq. (1.101) scale with the system size for $\delta \le 0$. Similarly, to the previous sections we can present solutions of Eq. (1.100) as

$$p_i = \frac{2\pi}{L}\left(-\frac{L+1}{2}+j-\omega_j\right), \qquad j = 1,\dots,\ell = L+\delta. \quad \text{(C.41)}$$

We use maximally allows set for $\mathbf{p}$, namely

$$\mathbf{p} = \{p_1,\dots p_\ell\} \quad \text{(C.42)}$$

and states $\mathbf{q}$ are parametrized by the set of $n = |\delta|$ holes as previously

$$\mathbf{q}^{a_1,\dots a_n} = \{q_1,\dots \hat{q}_{a_1},\dots \hat{q}_{a_n},\dots q_L\}. \quad \text{(C.43)}$$

Similar to Eq. (C.30) using Lemma (B.1) the overlap (1.101) can be presented as

$$|\langle \mathbf{p}|\mathbf{q}^{a_1,\dots a_n}\rangle|^2 = \frac{\prod_{i=1}^{\ell} e^{g(p_i)-g(q_i)} \prod_{i=1}^{n} e^{g(q_{a_i})-g(\pi)}}{\prod_{i=1}^{\ell}\left(1+\frac{2\pi}{L}\omega'(p_i)\right)} \frac{\prod_{i>j}^{\ell} \sin^2\frac{p_i-p_j}{2}}{\prod_{i>j}^{\ell} \sin^2\frac{q_i-q_j}{2}}$$

$$\times \frac{\prod_{i>j}^{\ell} \sin^2\frac{q_i-q_j}{2} \prod_{i>j}^{L} \sin^2\frac{q_j-q_i}{2}}{\prod_{i=1}^{\ell}\prod_{\substack{j=1\\ j\neq i}}^{L} \sin^2\frac{q_i-q_j}{2}} \times \prod_{i>j}^{n} \sin^2\frac{q_{a_i}-q_{a_j}}{2} \times \prod_{k=1}^{n} \frac{\prod_{i=1}^{\ell} \sin^2\frac{p_i-q_{a_k}}{2}}{\prod_{i\neq a_k}^{L} \sin^2\frac{q_i-q_{a_k}}{2}}. \quad \text{(C.44)}$$

This way, using formulas from the previous subsection (C.4), we see that overlap $|\langle \mathbf{p}|\mathbf{q}^{a_1,\dots a_n}\rangle|^2$ is identical to Eqs. (C.39), (C.40), upon the identification $\nu \to \omega$ and δ to be changed from by $\nu(\pi)-\nu(-\pi) \to \omega(\pi)-\omega(-\pi)$.

Appendix D

Regularization of the prefactor and power-like behavior

In this Appendix we describe a regularization of the divergent integral

$$\mathcal{A} = \log Z = \frac{1}{2}\int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dp\, \nu'(q)\nu'(k) \log\left|\sin\frac{q-k}{2}\right| \quad \text{(D.1)}$$

for the case of discontinuous $\nu(k)$. We use the regularization described in Eqs. (2.78) – (2.81) and find the asymptotics of this integral for large-times.

It is natural to divide the derivative of ν into two parts,

$$\nu'(k) = \nu_0'(k) + \nu_1'(k), \quad \text{(D.2)}$$

where

$$\nu_0'(k) = A'(k) + B'(k)s(k), \quad \nu_1'(k) = B(k)s'(k). \quad \text{(D.3)}$$

In the large t limit $\nu_0'(k)$ is a bounded function meanwhile $\nu_1'(k)$ becomes proportional to a δ-function. The double integral $\mathcal{A}$ can be presented as a sum of four parts

$$\mathcal{A} = \mathcal{A}_{00} + \mathcal{A}_{01} + \mathcal{A}_{10} + \mathcal{A}_{11}, \tag{D.4}$$

where

$$\mathcal{A}_{ij} = \frac{1}{2} \int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dp\, \nu_i'(q)\nu_j'(k) \log \left| \sin \frac{q-k}{2} \right|. \tag{D.5}$$

Note, only $\mathcal{A}_{11}$ part is responsible for the divergence of $\mathcal{A}$ at large t. The parts $\mathcal{A}_{00}$, $\mathcal{A}_{01}$ and $\mathcal{A}_{10}$ have non-singular limiting values at $t \to \infty$ which do not depend on regularization of $\nu(k)$. We have

$$\mathcal{A}_{00} \approx \frac{1}{2} \int_{-\pi}^{\pi} dq \int_{-\pi}^{\pi} dp\, [\nu'](q)[\nu'](k) \log \left| \sin \frac{q-k}{2} \right| \tag{D.6}$$

with

$$[\nu'](k) = A'(k) + B'(k) \operatorname{sign} \Phi'(k). \tag{D.7}$$

Due to $k \leftrightarrow q$ symmetry, we have $\mathcal{A}_{01} = \mathcal{A}_{10}$. In the limit $t \to \infty$, the function $\nu_1'(k)$ becomes a sum of two delta functions, and theqrefore

$$\begin{aligned} \mathcal{A}_{01} = \mathcal{A}_{10} \approx B_1 r_1 \int_{-\pi}^{\pi} dq\, [\nu'](q) \log \left| \sin \frac{q-q_1}{2} \right| \\ + B_2 r_2 \int_{-\pi}^{\pi} dq\, [\nu'](q) \log \left| \sin \frac{q-q_2}{2} \right|, \end{aligned} \tag{D.8}$$

where

$$B_i = B(q_i), \qquad r_i = \operatorname{sign}(\Phi''(q_i)) \tag{D.9}$$

To evaluate $\mathcal{A}_{11}$, we divide the integration region $[-\pi, \pi]$ into two pieces $\Lambda_1 = [-\pi, p]$ and $\Lambda_2 = (p, \pi]$, where point p lies between critical points $q_1 < p < q_2$. This way, the double integral $\mathcal{A}_{11}$ is divided into four parts

$$\mathcal{A}_{11} = a_{11} + a_{12} + a_{21} + a_{22}, \tag{D.10}$$

where

$$a_{ij} = \frac{1}{2} \int_{\Lambda_i} dq \int_{\Lambda_j} dk\, B(q)B(k)s'(q)s'(k) \log \left| \sin \frac{q-k}{2} \right|. \tag{D.11}$$

The integrals a_{21} and a_{12} have finite limits at $t\to\infty$

$$a_{12}=a_{21}\approx 2B_1B_2r_1r_2\log\sin\frac{q_2-q_1}{2}. \tag{D.12}$$

The remaining parts of $\mathcal{A}_{11}$ contain singularities. Let us show how they emerge in an example of a_{11}. It is natural to present a_{11} as a sum of two integrals (regular and singular)

$$a_{11}=a_{11}^{(r)}+a_{11}^{(s)}, \tag{D.13}$$

where

$$a_{11}^{(r)}=\frac{1}{2}\int\limits_{\Lambda_1}dq\int\limits_{\Lambda_1}dk\,B(q)B(k)s'(q)s'(k)\log\left|\frac{\sin\frac{q-k}{2}}{\Phi'(q)-\Phi'(k)}\right|, \tag{D.14}$$

$$a_{11}^{(s)}=\frac{1}{2}\int\limits_{\Lambda_1}dq\int\limits_{\Lambda_1}dk\,B(q)B(k)s'(q)s'(k)\log\left|\Phi'(q)-\Phi'(k)\right|. \tag{D.15}$$

The first integral can be found using L'Hôpital's rule

$$\begin{aligned}a_{11}^{(r)}=\frac{2r_1\cdot 2r_1}{2}&\int\limits_{\Lambda_1}dq\int\limits_{\Lambda_1}dk\,B(q)B(k)\delta(q-q_1)\delta(k-q_1)\\&\times\log\left|\frac{\sin\frac{q-k}{2}}{\Phi'(q)-\Phi'(k)}\right|=-2B_1^2\log|2\Phi''(q_1)|,\end{aligned} \tag{D.16}$$

where we used $r_1^2=1$. The second integral can be presented as

$$a_{11}^{(s)}=u_1+v_1\log\sqrt{t}, \tag{D.17}$$

where

$$u_1=\frac{1}{2}\int\limits_{\Lambda_1}dq\int\limits_{\Lambda_1}dk\,B(q)B(k)s'(q)s'(k)\log\left|\sqrt{t}\Phi'(q)-\sqrt{t}\Phi'(k)\right|, \tag{D.18}$$

$$v_1=-\frac{1}{2}\int\limits_{\Lambda_1}dq\int\limits_{\Lambda_1}dk\,B(q)B(k)s'(q)s'(k). \tag{D.19}$$

Performing rescaling of the integration variables one can persuade oneself that in under the last integrals $B(q)$ can be replaced to B_1, which leads to

$$v_1=-\frac{B_1^2}{2}\int\limits_{\Lambda_1}dq\int\limits_{\Lambda_1}dk\,s'(q)s'(k)=-2B_1^2. \tag{D.20}$$

Here we have used (2.81), and all the traces of the regularization has disappeared. With u_1 this will not be the same. Indeed, using $s(k) = f(\sqrt{t}\Phi'(k))$ and changing the variables of integration q and k by $\lambda = \sqrt{t}\Phi'(q)$ and $\mu = \sqrt{t}\Phi'(k)$, we get

$$u_1 = \frac{1}{2} \int\limits_{\tilde{\Lambda}_1} d\lambda \int\limits_{\tilde{\Lambda}_1} d\mu \, b(\lambda) b(\mu) f'(\lambda) f'(\mu) \log|\lambda - \mu|, \tag{D.21}$$

where the function $b(\lambda)$ is defined as

$$b(\sqrt{t}\Phi'(q)) = B(q) \tag{D.22}$$

and region $\tilde{\Lambda}_1$ is the segment $[\sqrt{t}\Phi'(-\pi), \sqrt{t}\Phi'(p)]$ which becomes the real line when t goes to infinity. Also $b(\lambda)$ goes to B_1 at $t \to \infty$. Theqrefore we get

$$u_1 \approx \frac{B_1^2}{2} \cdot \int\limits_{-\infty}^{\infty} d\lambda \int\limits_{-\infty}^{\infty} d\mu \, f'(\lambda) f'(\mu) \log|\lambda - \mu|. \tag{D.23}$$

Finally, using $B_1 = -\delta_1/2$, $B_2 = \delta_2/2$, $r_1 = -1$, and $r_2 = 1$, we obtain the following large t asymptotics of $\mathcal{A}$

$$\mathcal{A} \approx d_0 + d_1 \log \sqrt{t}, \tag{D.24}$$

where the constant d_1 is universal, i.e. it is independent of a regularizing function f

$$d_1 = -2(B_1^2 + B_2^2) = -\frac{1}{2}(\delta_1^2 + \delta_2^2) \tag{D.25}$$

and d_0 depends on a regularizing function f only in summands u_1 and u_2

$$d_0 = \mathcal{A}_{00} + 2\mathcal{A}_{01} + 2a_{12} + a_{11}^{(r)} + a_{22}^{(r)} + u_1 + u_2. \tag{D.26}$$

Appendix E

Green's function calculation

In this appendix we compute the thermodynamic limit of the Green's function $G(x, y, t)$ defined as

$$G^*(x, y, t) \equiv \sum_k \frac{\chi_k(x)\chi_k(y)}{(\chi_k, \chi_k)} e^{itE_k}, \qquad t \geq 0. \tag{E.1}$$

Here summation is taken over all solution of the spectrum condition (3.35). For a moment we focus on the case when bound states are absent in the spectrum. Using notations for χ_k in (3.33), the norm (3.38) and the phase (3.35). We present for one particular choice of the sign of the square root $\sqrt{1+(\operatorname{Re} b_k)^2}$

$$\frac{\chi_k(x)\chi_k(y)}{(\chi_k,\chi_k)} = \frac{1}{2(R+\delta'(k))}\operatorname{Re}\frac{\varphi_k(x)\bar{\psi}_k(y)}{a_k} - \frac{\operatorname{Re} Z_k(x,y)}{2(R+\delta'(k))\sqrt{1+(\operatorname{Re} b_k)^2}} \tag{E.2}$$

with

$$Z_k(x,y) = \frac{\psi_k(x)\psi_k(y)}{\bar{a}_k} + \frac{\operatorname{Re} b_k}{\bar{a}_k}\bar{\varphi}_k(x)\psi_k(y). \tag{E.3}$$

To evaluate the sum over k we first notice that the norm (3.38) can be presented as a derivative of the spectrum condition (3.35)

$$(\chi_k,\chi_k) = (\operatorname{Re} b_k + \sqrt{1+(\operatorname{Re} b_k)^2})\sqrt{1+(\operatorname{Re} b_k)^2}\frac{\partial_k[e^{2ikR+2i\delta(k)}-1]}{2i}. \tag{E.4}$$

Further we employ the residue theorem in the following form

$$\sum_k \frac{F(k)}{\partial_k S(k)} = \frac{1}{2\pi i}\oint_\gamma dk\frac{F(k)}{S(k)}, \tag{E.5}$$

where summation is over all solutions of the equation $S(k)=0$ and the contour γ runs around these values only and avoids any singularities of the function $F(k)$. This way we identically present

$$G^*(x,y,t) = \oint_\gamma \frac{dk}{2\pi}\frac{e^{itE_k}}{e^{2ikR+2i\delta_+(k)}-1}\left(\operatorname{Re}\frac{\varphi_k(x)\bar{\psi}_k(y)}{a_k} - \frac{\operatorname{Re} Z_k(x,y)}{\sqrt{1+(\operatorname{Re} b_k)^2}}\right) + (\delta_+\to\delta_-)\,, \tag{E.6}$$

where by $\delta_\pm$ we mean terms that are obtained by the flip of the sign $\sqrt{1+(\operatorname{Re} b_k)^2}$, specifically for the solutions of (3.35)

$$\frac{i\operatorname{Im} b_k + \sqrt{1+(\operatorname{Re} b_k)^2}}{\bar{a}_k} \equiv e^{-2i\delta(k)}. \tag{E.7}$$

The contour γ encompasses all solutions of $e^{2ikR+2i\delta(k)}=1$. We can present it as two contours below and above the real axes oriented in the positive and negative directions correspondingly. In the thermodynamic limit (with exponential accuracy) we notice that only the contour above the real line contributes, theqrefore we can present

$$G^*(x,y,t) = \int_0^\infty \frac{dk}{\pi} e^{itE_k} \mathrm{Re}\, \frac{\varphi_k(x)\bar{\psi}_k(y)}{a_k}. \tag{E.8}$$

Here we have taken into account that upon the summation $Z_k(x,y)$ terms cancel out. Identically we can present

$$G^*(x,y,t) = \int_{-\infty}^\infty \frac{dk}{2\pi} e^{itE_k} \frac{\varphi_k(x)\bar{\psi}_k(y)}{a_k}. \tag{E.9}$$

Notice that the function that we integrate can be analytically continued to the upper half plane. This allows us to write the general answer in the case when bound states are present in the system as

$$G^*(x,y,t) = \int_C \frac{dk}{2\pi} e^{itE_k} \frac{\varphi_k(x)\bar{\psi}_k(y)}{a_k}, \tag{E.10}$$

where the contour lies in the upper half above all positions of the bound states and connects $-\infty$ and $+\infty$.

Appendix F

Evaluation of $f_q^{(\alpha)}(t)$

In this appendix we demonstrate how to rigorously evaluate $f_q^{(\alpha)}(t)$ defined in (3.53). We focus on $f_q^{(0)}$, as the computation for $f_q^{(1)}(t)$ goes similarly. Namely, we are going to evaluate the thermodynamic limit of the discrete sum

$$f_q^{(0)}(t) = \sum_k \frac{(\Lambda_q, \chi_k)\chi_k(0)}{(\chi_k, \chi_k)} e^{itE_k}. \tag{F.1}$$

The main formal problem is that the overlap (Λ_q, χ_k) is singular on the real line, theqrefore the trick with the summation introduced in E requires small modifications in the part choosing the integration contours. More precisely to describe the singularity we assume that without loss of generality the eigenvalues of Λ_q and χ_k are different so the corresponding overlap could be found from

$$(k^2 - q^2)(\Lambda_q, \chi_k) = \Lambda_q'(0)\chi_k(0) - \int_{-R}^{0} dx \Lambda_q(x)(V_0(x) - V(x))\chi_k(x), \tag{F.2}$$

where we have used boundary conditions (3.13) and (3.14). This way, we present

$$(\Lambda_q, \chi_k) = \frac{\mathrm{Im}\left(e^{-i\delta(k)}\Xi_{q,k}^{\psi}\right)}{k^2 - q^2}, \tag{F.3}$$

$$\Xi_{q,k}^{\psi} = \Lambda_q'(0)\psi_k(0) - \int_{-\infty}^{0} dx \Lambda_q(x)(V_0(x) - V(x))\psi_k(x). \tag{F.4}$$

Notice that here we have replaced the lower integration boundary from $-R$ to $-\infty$, which is possible due to the finite range of the potential. Moreover, in this expression the dependence of the momenta k and q is smooth, so in particular the limit as $q \to k$ is well defined, contrary to the overall overlap, where special care has to be taken to the numerator. In particular, one can drop the quantization conditions for k and consider the limit $k \to q$

$$\Xi_{q,q}^{\psi} = \Lambda_q'(0)\psi_q(0) - \int_{-\infty}^{0} dx \Lambda_q(x)(V_0(x) - V(x))\psi_q(x). \tag{F.5}$$

To evaluate this expression we use the same trick as in (3.49), (3.51), which gives

$$\begin{aligned}\Xi_{q,q}^{\psi} = \Lambda_q'(-R)\psi_q(-R) = -\frac{qe^{iqR}}{\Phi_q(0)}\left(\bar{a}_q e^{iqR} - b_q e^{-iqR}\right) \\ = -\frac{q}{\Phi_q(0)}(\bar{a}_q e^{-2i\eta(q)} - b_q).\end{aligned} \tag{F.6}$$

Here at the last step we have used (3.43). For the direct proof of the result (F.6) from the definition (F.4) see G.

With all these notations the function $f_q^{(0)}(t)$ can be presented as

$$f_q^{(0)}(t) = \sum_k \frac{\mathrm{Im}(e^{-i\delta(k)}\Xi_{q,k})\mathrm{Im}(e^{-i\delta(k)}\psi_k(0))}{(k^2 - q^2)(\chi_k, \chi_k)} e^{itE_k}. \tag{F.7}$$

We are going to evaluate the sum in (F.7) in the thermodynamic limit by presenting it as a contour integral in a way similar to E

$$f_q^{(0)}(t) = \oint_\gamma \frac{dk}{\pi} \frac{e^{itk^2}}{e^{2ikR+2i\delta(k)} - 1} \times \frac{\mathrm{Im}(e^{-i\delta(k)}\Xi_{q,k})\mathrm{Im}\,(e^{-i\delta(k)}\psi_k(0))}{(k^2-q^2)(\mathrm{Re}\,b_k + \sqrt{1+(\mathrm{Re}\,b_k)^2})\sqrt{1+(\mathrm{Re}\,b_k)^2}}. \quad \text{(F.8)}$$

Here contour γ runs only around all positive solutions of the equation $e^{2ikR+2i\delta(k)} = 1$ and summation over two branches of the square root in (3.35) $\delta = \delta_\pm$ is assumed. The contour γ can be deformed into two contours above and below real line. But contrary to E we have to subtract contribution from the point $k = q$, theqrefore we can present $f_q^{(0)}(t)$ as

$$f_q^{(0)}(t) = \hat{f}_q^{(0)}(t) - f_q^{(0,+)}(t) + f_q^{(0,-)}(t), \quad \text{(F.9)}$$

where

$$\hat{f}_q^{(0)}(t) = -i\frac{\mathrm{Im}(e^{-i\delta(q)}\Xi^\psi_{q,q})\mathrm{Im}\,(e^{-i\delta(q)}\psi_q(0))}{q(\mathrm{Re}\,b_q + \sqrt{1+(\mathrm{Re}\,b_q)^2})\sqrt{1+(\mathrm{Re}\,b_q)^2}} \frac{e^{itq^2}}{e^{2i(\delta(q)-\eta(q))} - 1} \quad \text{(F.10)}$$

and

$$f_q^{(0,\pm)}(t) = \int_0^\infty \frac{dk}{\pi} \frac{e^{itk^2}}{e^{2i(k\pm i0)R+2i\delta(k)} - 1} \times \frac{\mathrm{Im}(e^{-i\delta(k)}\Xi_{q,k})\mathrm{Im}(e^{-i\delta(k)}\psi_k(0))}{((k\pm i0)^2-q^2)(\mathrm{Re}\,b_k + \sqrt{1+(\mathrm{Re}\,b_k)^2})\sqrt{1+(\mathrm{Re}\,b_k)^2}}. \quad \text{(F.11)}$$

In (F.10) we have used that the point q corresponds to the spectrum of the pre-quench spectrum (3.43). So far these transformations are exact. Further we address the large system size limit $R \to \infty$. In this limit the last term in (F.9) vanishes $f_q^{(0,-)} \to 0$, while $f_q^{(0,+)}(t)$ can be computed identically to G^* in E

$$f_q^{(0,+)}(t) = -\int_0^\infty \frac{dk}{\pi} \frac{\mathrm{Re}\left[\Xi^\varphi_{q,k}\partial_x^\alpha \bar{\psi}_k(0) a_k^{-1}\right] e^{itk^2}}{(k+i0)^2 - q^2}. \quad \text{(F.12)}$$

To compute the residue contribution $\hat{f}_q^{(0)}(t)$ we first use (3.35) to present

$$\frac{1}{e^{2i(\delta(q)-\eta(q))} - 1} = \frac{\sqrt{1+(\mathrm{Re}\,b_q)^2} + i\mathrm{Im}\,b_q + a_q e^{2i\eta(q)}}{-2i\mathrm{Im}[a_q e^{2i\eta(q)} + b_q]}, \quad \text{(F.13)}$$

and then perform summation over all branches of the square root to obtain

$$\hat{f}_q^{(0)}(t) = -\frac{e^{itq^2}}{2q} \times \frac{\mathrm{Re}\left[\Xi^{\psi}_{q,q}\partial_x^\alpha\psi_q(0)\bar{a}_q^{-1}\right] - (a_q e^{2i\eta(q)} - \bar{b}_q)\mathrm{Re}\left[\Xi^{\varphi}_{q,q}\partial_x^\alpha\bar{\psi}_q(0)a_q^{-1}\right]}{\mathrm{Im}[a_q e^{2i\eta(q)} - \bar{b}_q]}. \quad (F.14)$$

Here we have introduced

$$\Xi^{\varphi}_{q,k} \equiv a_k\Xi^{\psi}_{q,k} + b_k\bar{\Xi}^{\psi}_{q,k} = \Lambda_q'(0)\varphi_k(0) - \int\limits_{-\infty}^{0} dx\Lambda_q(x)(V_0(x) - V(x))\varphi_k(x), \quad (F.15)$$

which coincides with $\Xi_{q,k}$ in the main text (see (3.48)). The diagonal component can be obtained from (F.6),

$$\Xi^{\varphi}_{q,q} = -\frac{q}{\bar{\Phi}_q(0)}, \quad (F.16)$$

which allows us to significantly simplify expression for $\hat{f}_q^{(0)}$. Overall, for $f_q^{(\alpha)}$ we obtain the following expression

$$f_q^{(\alpha)}(t) = \frac{\partial_x^\alpha\psi_q(0)e^{itq^2}}{2i\bar{a}_q\Phi_q(0)} + \int\limits_0^\infty \frac{dk}{\pi}\mathrm{Re}\left[\frac{\bar{\Xi}^{\varphi}_{q,k}\partial_x^\alpha\psi_k(0)}{\bar{a}_k}\right]\frac{e^{itk^2}}{(k+i0)^2 - q^2}. \quad (F.17)$$

Notice that extending the integration over k to the negative values we can also present

$$f_q^{(\alpha)}(t) = \int\limits_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{\Xi^{\varphi}_{q,k}\partial_x^\alpha\bar{\psi}_k(0)}{a_k}\frac{e^{itk^2}}{(k+i0)^2 - q^2}. \quad (F.18)$$

Now let us discuss on how to account for the bound states. As we discussed in Section 3.2 the bound states' wave function can be understood as the Jost functions analytically continued to the upper half plane and evaluated at the purely imaginary momenta $\chi_n^{\mathrm{bound}}(x) = \varphi_{i\varkappa_n}(x)$. The contributions from the bound states modify (F.18) as follows

$$f_q^{(\alpha)}(t) = \sum_{n=1}^{N^{\mathrm{b}}}\frac{(\Lambda_q, \varphi_{i\varkappa_n})\partial_x^\alpha\varphi_{i\varkappa_n}(0)}{(\varphi_{i\varkappa_n}, \varphi_{i\varkappa_n})}e^{-it\varkappa_n^2} + \int\limits_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{\Xi^{\varphi}_{q,k}\partial_x^\alpha\bar{\psi}_k(0)}{a_k}\frac{e^{itk^2}}{(k+i0)^2 - q^2}. \quad (F.19)$$

Using the normalization (3.41) and the relation $\varphi_{i\varkappa} = b_\varkappa \bar{\psi}_{i\varkappa}$, we see that we can present $f_q^{(\alpha)}$ in the following way

$$f_q^{(\alpha)}(t) = \int\limits_C \frac{dk}{2\pi} \frac{\Xi^{\varphi}_{q,k} \partial_x^\alpha \bar{\psi}_k(0)}{a_k} \frac{e^{itk^2}}{k^2 - q^2}, \tag{F.20}$$

where the contour C runs from $-\infty$ to $+\infty$ and lies in the upper-half plane above all zeroes of a_k. In this form this expression coincides with (3.53) obtained directly by going into the thermodynamic limit on the level of the Green's function.

Appendix G

Evaluation of $\Xi^{\varphi}_{q,q}$

In this appendix, using definition (F.15)

$$\Xi^{\varphi}_{q,q} \equiv \Xi_{q,q} = \Lambda'_q(0)\varphi_q(0) - \int\limits_{-\infty}^{0} dx \Lambda_q(x)(V_0(x) - V(x))\varphi_k(x), \tag{G.1}$$

we prove that

$$\Xi_{q,q} = -\frac{q}{\bar{\Phi}_q(0)}, \tag{G.2}$$

which is the statement (F.16). Taking into account that $\Xi_{q,-q} = \bar{\Xi}_{q,q}$ we obtain (3.51). Finally, the statement (F.6) can be considered as a sequence of these two results and the relation $\Xi_{q,k} = a_k \Xi^{\psi}_{q,k} + b_k \bar{\Xi}^{\psi}_{q,k}$.

We start the proof by noticing that from the integral presentation for the Jost solutions Φ_q

$$\Phi_q(x) = e^{-iqx} + \int\limits_{-\infty}^{x} \frac{\sin(q(x-y))}{q} V_0(y)\Phi_q(y)dy, \tag{G.3}$$

one can immediately obtain

$$\Phi_q(0) = 1 - \int\limits_{-\infty}^{0} \frac{\sin(qy)}{q} V_0(y)\Phi_q(y)dy, \tag{G.4}$$

$$\Phi'_q(0) = -iq + \int\limits_{-\infty}^{0} \cos(qy) V_0(y) \Phi_q(y) dy. \tag{G.5}$$

So

$$\Phi'_q(0) + iq\Phi_q(0) = \int\limits_{-\infty}^{0} e^{-iqy} V_0(y) \Phi_q(y) dy \tag{G.6}$$

and

$$\Phi'_q(0) - iq\Phi_q(0) + 2iq = \int\limits_{-\infty}^{0} e^{iqy} V_0(y) \Phi_k(y) dy. \tag{G.7}$$

Invoking notation for the hard-wall wave function (3.42)

$$\Lambda_q(x) = \mathrm{Im} \frac{\Phi_q(x)}{\Phi_q(0)}, \tag{G.8}$$

we see that (G.1) can be written as

$$\Xi_{q,q} = \frac{1}{2i} \left(\frac{\Phi'_q(0)\varphi_q(0) - I_1}{\Phi_q(0)} - \frac{\bar{\Phi}'_q(0)\varphi_q(0) - \bar{I}_2}{\bar{\Phi}_q(0)} \right), \tag{G.9}$$

where

$$I_1 = \int\limits_{-\infty}^{0} dx \Phi_q(x)(V_0(x) - V(x))\varphi_q(x), \tag{G.10}$$

$$I_2 = \int\limits_{-\infty}^{0} dx \Phi_q(x)(V_0(x) - V(x))\bar{\varphi}_q(x). \tag{G.11}$$

Using integral presentation for $\varphi_q(x)$ in the first term and for $\Phi_q(x)$ in the second term we obtain

$$\begin{aligned} I_1 = \int\limits_{-\infty}^{0} dx \Phi_q(x) V_0(x) e^{-iqx} - \int\limits_{-\infty}^{0} dx e^{-iqx} V(x) \varphi_q(x) \\ + \int\limits_{-\infty}^{0} dx \int\limits_{-\infty}^{x} dy \frac{\sin(q(x-y))}{q} \Phi_q(x) V_0(x) V(y) \varphi_q(y) \\ - \int\limits_{-\infty}^{0} dx \int\limits_{-\infty}^{x} dy \frac{\sin(q(x-y))}{q} \Phi_q(y) V_0(y) V(x) \varphi_q(x). \end{aligned} \tag{G.12}$$

Changing variables in the last two integrals, we arrive at

$$I_1 = \int\limits_{-\infty}^{0} dx \Phi_q(x) V_0(x) e^{-iqx} - \int\limits_{-\infty}^{0} dx e^{-iqx} V(x) \varphi_q(x)$$
$$+ \int\limits_{-\infty}^{0} dx \int\limits_{-\infty}^{0} dy \frac{\sin(q(x-y))}{q} \Phi_q(x) V_0(x) V(y) \varphi_q(y). \quad \text{(G.13)}$$

Presenting sine in the exponential form and substituting right hand sides of (G.6) and (G.7) we obtain

$$I_1 = \Phi_q'(0)\varphi_q(0) - \varphi_q'(0)\Phi_q(0). \quad \text{(G.14)}$$

Similarly we can compute I_2

$$I_2 = 2iq + \Phi_q'(0)\bar{\varphi}_q(0) - \bar{\varphi}_q'(0)\Phi_q(0). \quad \text{(G.15)}$$

Substitution of I_1 and I_2 into (G.9) finishes the proof.

Appendix H

Kernels and scattering data for specific potentials

H.1 Single delta potential

In this appendix we present explicit formulas for the scattering data and the FCS for the quench situation that corresponds to $V(x) = g\delta(x)$, $V_0(x) = 0$.

The Jost functions can easily found from the integral presernations (3.4) and (3.5)

$$\psi_k(x) = e^{-ikx} - \frac{g}{k}\theta(-x)\sin(kx), \quad \text{(H.1)}$$

$$\varphi_k(x) = e^{-ikx} + \frac{g}{k}\theta(x)\sin(kx), \quad \text{(H.2)}$$

where $\theta(x)$ is Heaviside step function. The scattering data can be immediately read off from this presentation

$$a_k = 1 - \frac{g}{2ik}, \qquad b_k = \frac{g}{2ik}, \quad \text{(H.3)}$$

$$T(E) = \frac{1}{|a_k|^2} = \frac{k^2}{k^2 + g^2/4} = \frac{E}{E + g^2/4}. \tag{H.4}$$

To describe bound states we introduce $\varkappa = |g|/2$. If $g < 0$ the bound state corresponds to the zero of a_k at the momentum $k = i\varkappa$. The corresponding wave function reads

$$\varphi_{i\varkappa}(x) = e^{-\varkappa|x|}. \tag{H.5}$$

The Jost and hard-wall wave functions corresponding to the initial potential $V_0(x) = 0$ are

$$\Phi_q(x) = e^{-iqx}, \qquad \Lambda_q(x) = \operatorname{Im}\Phi_q(x) = -\sin qx. \tag{H.6}$$

This leads to $\Xi_{q,k} = \Lambda_q'(0)\varphi_k(0) = -q$. Using presentation (3.55) we obtain

$$f_q^{(1)}(t) = \frac{1}{2} q e^{itq^2}, \tag{H.7}$$

$$f_q^{(0)}(t) = -\frac{1}{2}\frac{qe^{itq^2}}{iq + g/2} - \theta(-g)\frac{q\varkappa e^{-it\varkappa^2}}{\varkappa^2 + q^2} + qE_\varkappa(q), \tag{H.8}$$

where

$$E_\varkappa(q) = \int\limits_0^\infty \frac{dp}{\pi}\frac{p^2 e^{itp^2}}{(p^2+\varkappa^2)((p+i0)^2 - q^2)} = \frac{\varkappa \bar{h}_\varkappa(t)}{2(q^2+\varkappa^2)} - \frac{iqh_q(t)}{2(q^2+\varkappa^2)}. \tag{H.9}$$

and

$$h_q(t) = e^{itq^2}\left[1 - \operatorname{Erf}\left(qe^{i\pi/4}\sqrt{t}\right)\right]. \tag{H.10}$$

The FCS can be written as

$$\mathcal{F}(\lambda, t) = \det\left(1 + \frac{e^\lambda - 1}{\pi}\rho(q)X_0(q,q') + \frac{e^\lambda - 1}{\pi}\rho(q)X_1(q,q')\right), \tag{H.11}$$

where

$$X_0(q,q') = qq'\left(\frac{q}{\varkappa^2+q^2} + \frac{q'}{\varkappa^2+q'^2}\right)\frac{\sin\left[t(q^2-q'^2)/2\right]}{q^2-q'^2}, \tag{H.12}$$

$$\begin{aligned} X_1(q,q') = &-2qq'\operatorname{Im}\left(e^{-it(q^2+q'^2)/2}\frac{e(q)-e(q')}{q^2-q'^2}\right) \\ &+ \frac{qq'}{(\varkappa^2+q^2)(\varkappa^2+q'^2)}\left\{\varkappa \operatorname{Re}\left(e^{it(q^2+q'^2)/2}h_\varkappa(t)\right)\right. \\ &\left. -\frac{g}{2}\cos\left[t(q^2-q'^2)/2\right] - 2\theta(-g)\varkappa\cos\left[t(q^2+q'^2+2\varkappa^2)/2\right]\right\}, \end{aligned} \tag{H.13}$$

and

$$e(q) = \frac{q h_q(t)}{2(q^2 + \varkappa^2)}. \tag{H.14}$$

In the notations of (3.61) $K_0 = \rho(e^\lambda - 1)X_0$ and $\delta K = \rho(e^\lambda - 1)X_1$.

The propagation emerging from a step initial distribution formally corresponds to $V_0(x) = 0$ for $x < 0$ and $V(x) = 0$. The corresponding FCS can be obtained from the above formulas by simply sending $g \to 0$. The corresponding kernels simplify as follows

$$X_0(q, q') = (q + q') \frac{\sin\left[t(q^2 - q'^2)/2\right]}{q^2 - q'^2}, \tag{H.15}$$

$$X_1(q, q') = -\mathrm{Im}\left(e^{-it(q^2+q'^2)/2} \frac{q' h_q(t) - q h_{q'}(t)}{q^2 - q'^2}\right). \tag{H.16}$$

H.2 Reflectionless potential

In this appendix we consider an example of perfect lead attachment, i.e. $V_0(x) = V(x)$, $x < 0$, for the reflectionless potential

$$V(x) = -\frac{2}{\cosh^2 x}. \tag{H.17}$$

The corresponding Jost solutions are

$$\psi_k(x) = e^{-ikx}\left(1 + \frac{2i}{k - i}\frac{1}{e^{2x} + 1}\right), \tag{H.18}$$

$$\varphi_k(x) = \bar{\psi}_k(-x) = e^{-ikx}\left(1 - \frac{2i}{k + i}\frac{1}{e^{-2x} + 1}\right) = \frac{k - i}{k + i}\psi_k(x), \tag{H.19}$$

which lead to the following scattering data

$$a_k = \frac{k - i}{k + i}, \qquad b_k = 0. \tag{H.20}$$

This potential has one bound state corresponding to the zero of a_k at $k = i$:

$$\chi_1^{\mathrm{b}}(x) = \varphi_{k=i}(x) = \frac{1}{2\cosh x}. \tag{H.21}$$

The hard-wall wave functions defined in (3.42) are given by

$$\Lambda_q(x) = -\frac{q \sin qx + \tanh x \cos qx}{q}. \tag{H.22}$$

Theqrefore $\Xi_{q,k}$ in (3.48) becomes

$$\Xi_{q,k} = \Lambda'_q(0)\varphi_k(0) = -\frac{1+q^2}{q}\cdot\frac{k}{k+i}. \tag{H.23}$$

Using definitions (3.55) we arrive at

$$f_q^{(1)}(t) = -\frac{1+q^2}{2q}e^{itq^2}, \tag{H.24}$$

$$f_q^{(0)}(t) = \frac{e^{-it}}{2q} + \frac{e^{itq^2}}{2i} - \frac{1+q^2}{q}E_1(q) \tag{H.25}$$

where $E_\varkappa$ is defined in (H.9). Substituting these expression into (3.79) we arrive at (3.86).

H.3 Double delta barrier

The double delta barrier potential is given by

$$V(x) = g_1\delta(x-d_1) + g_2\delta(x-d_2), \tag{H.26}$$

where we assume that $d_2 > 0 > d_1$. The Jost solutions for this potential can be found via the integral presentations (3.4) and (3.5)

$$\begin{aligned}\psi_k(x) = e^{-ikx} - \theta(d_1-x)\frac{\sin(k(x-d_1))}{k}g_1\psi_k(d_1)\\ - \theta(d_2-x)\frac{\sin(k(x-d_2))}{k}g_2\psi_k(d_2),\end{aligned} \tag{H.27}$$

$$\begin{aligned}\varphi_k(x) = e^{-ikx} + \theta(x-d_1)\frac{\sin(k(x-d_1))}{k}g_1\varphi_k(d_1)\\ + \theta(x-d_2)\frac{\sin(k(x-d_2))}{k}g_2\varphi_k(d_2),\end{aligned} \tag{H.28}$$

where

$$\psi_k(d_1) = e^{-ikd_1}\left(1+\frac{g_2}{2ik}\right) - \frac{g_2}{2ik}e^{ik(d_1-2d_2)}, \qquad \psi_k(d_2) = e^{-ikd_2}, \tag{H.29}$$

$$\varphi_k(d_1) = e^{-ikd_1}, \qquad \varphi_k(d_2) = e^{-ikd_2}\left(1-\frac{g_1}{2ik}\right) + \frac{g_1}{2ik}e^{ik(d_2-2d_1)}. \tag{H.30}$$

The scattering data follows from (3.6)

$$a_k = \frac{g_1 g_2 e^{-2ik(d_1-d_2)} + (2k+ig_1)(2k+ig_2)}{4k^2}, \tag{H.31}$$

$$b_k = \frac{g_2 e^{-2id_2k}(g_1-2ik) - g_1 e^{-2id_1k}(g_2+2ik)}{4k^2}. \tag{H.32}$$

If we were interested only in the scattering data we could easily found them using results of H.1. Indeed, for any potential that can be presented as a disjoint sum i.e. $V(x) = V_1(x) + V_2(x)$ with $V_1(x) = 0$ for $x > x_1$ and $V_2(x) = 0$ for $x < x_2$, where $x_1 < x_2$, the transfer matrix reads

$$\mathcal{T} = \mathcal{T}_1\mathcal{T}_2, \tag{H.33}$$

where $\mathcal{T}_j$ is the transfer matrix for V_j. This statement follows immediately from the relation of $\mathcal{T}_j$ to the corresponding Jost solutions ψ_j and φ_j, namely,

$$\begin{pmatrix}\varphi_1\\ \bar{\varphi}_1\end{pmatrix} = \mathcal{T}_1\begin{pmatrix}\psi_1\\ \bar{\psi}_1\end{pmatrix} = \mathcal{T}_1\begin{pmatrix}\varphi_2\\ \bar{\varphi}_2\end{pmatrix} = \mathcal{T}_1\mathcal{T}_2\begin{pmatrix}\psi_2\\ \bar{\psi}_2\end{pmatrix}. \tag{H.34}$$

Further, taking into account that the transfer matrix $\tilde{\mathcal{T}}$ for the shifted potential $\tilde{V}(x) = V(x-d)$ is related to $\mathcal{T}$ by conjugation with a diagonal matrix

$$\tilde{\mathcal{T}} = \mathcal{T}(d) = \begin{pmatrix} a_k & b_k e^{-2ikd}\\ \bar{b}_k e^{2ikd} & \bar{a}_k\end{pmatrix}, \tag{H.35}$$

the scattering data (H.31) and (H.32) for the potential (H.26) is recovered from

$$\mathcal{T} = \mathcal{T}_{g_1}(d_1)\mathcal{T}_{g_2}(d_2), \qquad \mathcal{T}_g(0) = \begin{pmatrix} 1-\frac{g}{2ik} & \frac{g}{2ik}\\ -\frac{g}{2ik} & 1+\frac{g}{2ik}\end{pmatrix}, \tag{H.36}$$

where for $\mathcal{T}_g(0)$ we used (H.3). The bound states correspond to zeroes of a_k in the upper half plane of k. For negative coupling constants g_1 and g_2 we have two bound states if

$$d_2 - d_1 > \frac{1}{|g_1|} + \frac{1}{|g_2|}, \tag{H.37}$$

and one otherwise.

The symmetric potential corresponds to $g_1 = g_2 = g$, $d_2 = -d_1 = d/2$. We introduce notations

$$k = i\varkappa, \qquad u = 2\varkappa/|g| > 0, \qquad D = |g|d, \tag{H.38}$$

so that the quantity $\varkappa$ describes the "momentum" of the bound state. The condition (H.37) now reads $D > 2$ (see also discussion around equation (3.89)). The current and the kernel in this case are obtained by the numerical integration of the corresponding expressions constructed via $f_q^{(\alpha)}(t)$ in (3.55). For the case when $V_0(x) = 0$ we have (H.6). Hence

$$\Xi_{qk} = \Lambda_q'(0)\varphi_k(0) + \int_{-\infty}^{0} dx \Lambda_q(x) V(x) \varphi_k(x)$$
$$= -q - g e^{ikd/2} \left(\frac{q}{k} \sin \frac{kd}{2} - \sin \frac{qd}{2} \right), \quad \text{(H.39)}$$

and

$$f_q^{(1)}(t) = B_{2,q}^{(1)} e^{-it\varkappa_2^2} + F_q^{(1)} e^{itq^2} + I_q^{(1)}(t), \tag{H.40}$$

$$f_q^{(0)}(t) = B_{1,q}^{(0)} e^{-it\varkappa_1^2} + F_q^{(0)} e^{itq^2} + I_q^{(0)}(t), \tag{H.41}$$

$$B_{n,q}^{(\alpha)} = \frac{i\Xi_{q,i\varkappa_n} \partial_x^\alpha \bar{\psi}_{i\varkappa_n}(0)}{a'_{i\varkappa_n}(\varkappa_n^2 + q^2)}, \qquad F_q^{(\alpha)} = -i \frac{\partial_x^\alpha \psi_q(0)}{2a_{-q}}, \tag{H.42}$$

$$a'_{i\varkappa_j} = \left.\frac{da}{dk}\right|_{k=i\varkappa_j} = -\frac{2i}{|g|} \frac{(u_j - 1)(D(u_j - 1) + 2)}{u_j^2}, \tag{H.43}$$

$$\bar{\psi}_{i\varkappa_1}(0) = 2 - 2/u_1, \qquad \bar{\psi}_{i\varkappa_2}(0) = \partial_x \bar{\psi}_{i\varkappa_1}(0) = 0, \tag{H.44}$$

$$\partial_x \bar{\psi}_{i\varkappa_2}(0) = (1 - u_2)|g|. \tag{H.45}$$

The integrals are given by

$$I_q^{(\alpha)}(t) = \int_0^\infty \frac{dk}{\pi} \Omega_{q,k}^{(\alpha)} \frac{e^{itk^2}}{(k + i0)^2 - q^2}, \tag{H.46}$$

with

$$\Omega_{q,k}^{(0)} = \frac{2k^2(-q + g \cos kd/2 \sin qd/2)}{g^2 + 2k^2 + g^2 \cos kd - 2gk \sin kd}, \tag{H.47}$$

$$\Omega_{q,k}^{(1)} = -\frac{2gk^3 \sin kd/2 \sin qd/2}{g^2 + 2k^2 - g^2 \cos kd + 2gk \sin kd}. \tag{H.48}$$

The asymptotic behavior of the integrals $I_q^{(\alpha)}(t)$ at large t is governed by expansions of the integrands at $k = 0$

$$\Omega_{q,k}^{(0)} = \frac{k^2}{g^2}(-q + g \sin qd/2) + O(k^4), \quad \Omega_{q,k}^{(1)} = -\frac{2k^2 gd \sin qd/2}{(2 + gd)^2} + O(k^4). \tag{H.49}$$

The formula for $\Omega^{(1)}_{q,k}$ is valid for $D = -gd \neq 2$. The asymptotic behavior of $\Omega^{(1)}_{q,k}$ for $D = 2$ and small k is

$$\Omega^{(1)}_{q,k} = \frac{4}{d^2}\sin\frac{qd}{2} + \frac{k^2}{18}\sin\frac{qd}{2} + O(k^4). \tag{H.50}$$

Theqrefore, the integrals have the following decaying behavior for large t

$$I^{(\alpha)}_q(t) \sim t^{-\frac{3}{2}} \quad \text{for} \quad D \neq 2, \quad \text{and} \quad I^{(\alpha)}_q(t) \sim t^{-\frac{3}{2}+\alpha} \quad \text{for} \quad D = 2. \tag{H.51}$$

This demonstrates that they do not affect the leading contribution in the asymptotic current (3.85). If the potential has two bound states than there is an oscillatory part of the current with the amplitude of oscillations given by (3.84)

$$A_{12} = -\frac{4}{\pi}\int_0^\infty dq\rho(q)B^{(1)}_{2,q}B^{(0)}_{1,q}. \tag{H.52}$$

Finally, the leading contribution to the current for large t consists of constant Landauer–Büttiker current and an oscillating current (if there are two bound states).

Appendix I
List of publications of the PhD candidate

1. O. Gamayun, N. Iorgov and Y. Zhuravlev, *Effective free-fermionic form factors and the XY spin chain*, SciPost Physics **10**(3) (2021), **[Q1]**
   doi: 10.21468/scipostphys.10.3.070 .

2. Y. Zhuravlev, E. Naichuk, N. Iorgov and O. Gamayun, *Large-time and long-distance asymptotics of the thermal correlators of the impenetrable anyonic lattice gas*, Physical Review B **105**(8) (2022), **[Q1]**
   doi: 10.1103/physrevb.105.085145 .

3. O. Gamayun, Y. Zhuravlev and N. Iorgov, *On Landauer–Büttiker formalism from a quantum quench*, Journal of Physics A: Mathematical and Theoretical **56**(20), 205203 (2023), **[Q2]**
   doi:10.1088/1751-8121/accabf .

Appendix J

Approbation of the results of the dissertation

1. Yurii Zhuravlov, "Effective free-fermionic form factors on a lattice and XY quantum chain", XI CONFERENCE OF YOUNG SCIENTISTS "PROBLEMS OF THEORETICAL PHYSICS", 21-23 December 2020,

   `https://indico.bitp.kiev.ua/event/7/contributions/193/`

2. Yurii Zhuravlov, "Large time and long distance asymptotics of the thermal correlators of the impenetrable anyonic lattice gas", XII CONFERENCE OF YOUNG SCIENTISTS "PROBLEMS OF THEORETICAL PHYSICS", 21-22 December 2021,

   `https://indico.bitp.kiev.ua/event/8/contributions/228/`

3. Yurii Zhuravlov, "On Landauer–Büttiker formalism from a quantum quench", XIII CONFERENCE OF YOUNG SCIENTISTS "PROBLEMS OF THEORETICAL PHYSICS", 21 December 2022,

   `https://indico.bitp.kiev.ua/event/10/contributions/245/`